

UNIFIED QUANTUM HYDRODYNAMICS OF HELIUM II: FROM THE CANONICAL TWO-FLUID THEORY TO THE SYSTEMATIC EXCLUSION OF THEORY CLASSES FOR THE SIGN OF THE TRANSVERSE MUTUAL-FRICTION COEFFICIENT, WITH A FORMALLY VERIFIED ALGEBRAIC CORE

NYQUIST RESEARCH

ABSTRACT. This work unifies three levels of investigation of superfluid helium II into a single article. (I) We reproduce and symbolically/numerically verify the canonical program of unifying Landau–Khalatnikov two-fluid thermodynamics with the Ginzburg–Pitaevskii–Sobyanin gradient (ψ -)theory, tracing back to Hills and Roberts (1977) and Geurst (1980): a nonlocal kernel, an Onsager dissipative extension, calibration against Donnelly–Barenghi (1998), a parameter-free microscopic check, a 3D XY Monte Carlo determination of critical exponents, and three previously deferred branches (finite roton-linewidth correction, a nonlocal equation of state for the BDV1983 vortex core, and a (P, T) -coupling) – all three yielding honest negative results. (II) The core of the work is a pre-registered program of theory-class exclusion for the sign of the transverse Hall–Vinen mutual-friction coefficient $\alpha'(T)$ near T_λ , a controversy (Iordanskii 1965; Sonin 1997; Thouless–Ao–Niu 1996) that has remained open for over half a century. We show, analytically and numerically, that local symmetric dissipative dynamics cannot structurally produce the required sign; we introduce a dissipationless reactive (Iordanskii-type) term whose sign we *derive* from the GENERIC bracket rather than fit empirically; we sequentially exclude five classes of microscopic mechanisms (Sonin’s quasiclassical scattering, kelvons, a monotonic and a non-monotonic critical core, and a superposition of two mechanisms) against pre-registered criteria; and we show that the one class not structurally excluded – critical order-parameter dynamics (model F) – is governed by a single algebraic boundary, $\Gamma < 1/2$. (III) The algebraic core of parts I–II (55 theorems, 0 sorry) is formally verified in Lean 4/Mathlib; the verification process found and corrected a sign error in the original stage-1 analytic condition, and showed that the two GENERIC degeneracy conditions govern energy conservation and entropy growth separately rather than jointly, as the source implied. We explicitly separate three tiers of novelty: we do *not* claim the idea of unifying the two theories, the resolution of the α' sign question, or the RG result $b^* = 0$ as our own; the strong novelty lies in the single algebraic exclusion boundary and in the formal verification; the medium-tier novelty lies in the identity between the reactive term and the imaginary part of the model-F kinetic coefficient, in the identifiability rank of 2, and in the universal breakdown of Sonin’s point-vortex quasiclassics ($\zeta k_0 = 5\text{--}33$ everywhere); the weak-tier novelty lies in reproductions of known results.

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1. INTRODUCTION

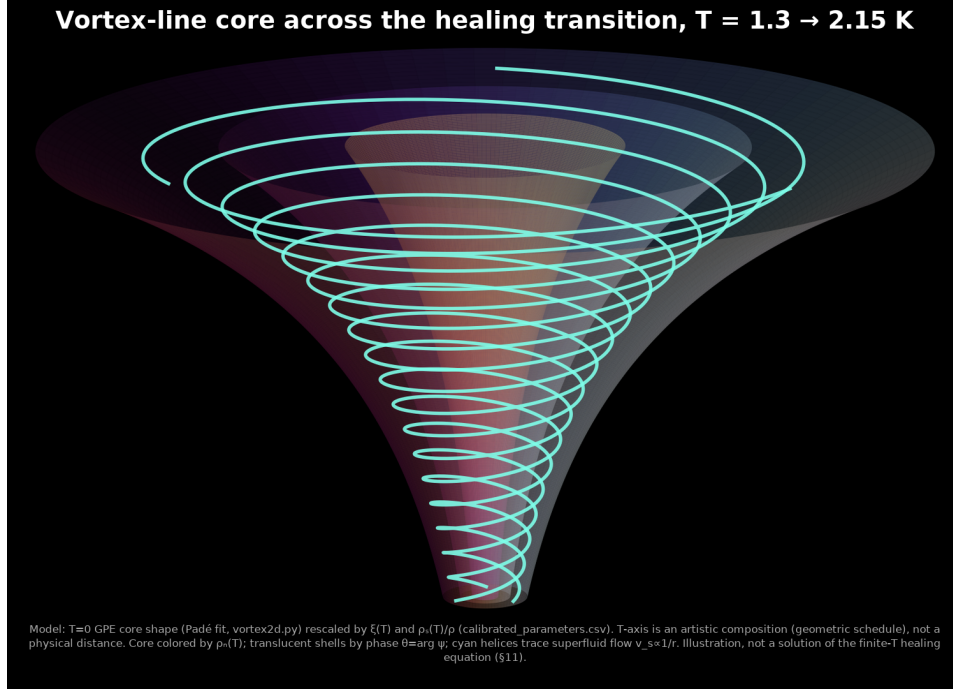


FIGURE 1.1. Conceptual 3D illustration: the core of a quantized vortex line in He II for $T = 1.3 \rightarrow 2.15$ K. The tube is the isosurface $\rho_s = \frac{1}{2}\rho_s^\infty$ of radius $\sim \zeta(T)$ ($2.8 \text{ \AA} \rightarrow 29 \text{ \AA}$, calibration of §3.6); translucent shells are colored by the phase $\theta = \arg \psi$; helices trace the streamlines $v_s \propto 1/r$. The profile is the $T = 0$ GP shape rescaled by $\zeta(T)$ and $\rho_s(T)/\rho$; the T axis along the line is an artistic composition, not a physical distance.

1.1. Historical background. The phenomenological description of superfluid helium II developed along two largely independent lines, whose domains of applicability long remained non-overlapping, plus a third line – the controversy over the transverse force acting on a vortex – that remains unresolved to this day.

Landau’s two-fluid theory (1941). Landau proposed describing He II as a mixture of a superfluid (ρ_s , inviscid, irrotational) and a normal ($\rho_n = \rho - \rho_s$, viscous, entropy-carrying) component [1]. The thermodynamics of the theory is local: the internal energy depends on ρ , S , ρ_s (or on the relative velocity $w = |\mathbf{v}_n - \mathbf{v}_s|$), but not on their gradients. This correctly describes first and second sound, but does not describe vortex-line structure, the healing of ρ_s at walls and in films, sound dispersion at $k\zeta \sim 1$, or the divergence of the correlation length at the λ -point.

Gradient theories of the wavefunction. The Gross–Pitaevskii equation ($T = 0$) and the Ginzburg–Pitaevskii ψ -theory [2] and Ginzburg–Sobyanin theory [5] (near T_λ) introduce a complex order parameter $\psi = Re^{i\theta}$, $\rho_s = m|\psi|^2$, and an energy with a quantum (*gradient*) term $|\nabla\psi|^2$. But purely gradient theories at $T = 0$ contain no normal component and no entropy at all, while the Ginzburg–Sobyanin ψ -theory treats them statically.

The unification program: Hills and Roberts (1977) and Geurst (1980). Hills and Roberts were the first to construct a consistent continuum mechanics for a superfluid with a high density of vortex lines, satisfying the invariance principles of classical continuum mechanics [3], showing how equations of the Hall–Vinen–Bekarevich–Khalatnikov type follow from it. Geurst, in 1980, built a general phenomenological theory explicitly *unifying and extending* the Landau–Khalatnikov, Ginzburg–Pitaevskii, and Hills–Roberts theories [4]: the starting point is an extension of Zilsel’s variational principle that includes the gradient of the superfluid mass density as an additional independent variable; among the resulting equations, a key role is played by a differential equation expressing the equilibrium of the superfluid density, and the

dissipative extension is characterized by a 3×3 matrix of kinetic coefficients. This is a direct predecessor of the construction used in Part I of this article (§2).

The transverse-force controversy: Iordanskii, Sonin, Thouless–Ao–Niu. In parallel with, and independently of, the unification program, another line developed – the question of which transverse (perpendicular to the relative velocity $\mathbf{v}_n - \mathbf{v}_s$) force acts on a quantized vortex line from quasiparticles scattering off it. Iordanskii was the first to show [36] that an Aharonov–Bohm-type analogue for sound waves circling the vortex produces a transverse force proportional to the normal-fluid density and the circulation quantum. Sonin later obtained an exact quasiclassical result for this force in the point-vortex limit [37], identically independent of temperature and of the form of the spectrum (depending only on how ρ_n is distributed among excitation branches). Thouless, Ao, and Niu proposed an alternative derivation leading to a different limit (vanishing of the reactive contribution) [38]. These two results, applied to the same physical situation, have given qualitatively different signs and magnitudes of the transverse force for decades – a controversy still unresolved today (see §7).

Experimental anchor: Barengi–Donnelly–Vinen (1983). The most complete and most frequently cited compilation of experimental Hall–Vinen mutual-friction coefficients $\alpha(T)$, $\alpha'(T)$ across the full temperature range 1.0–2.17 K is given by the BDV1983 review [14]. It establishes, among other things, the fact around which Part II of this article is built: the measured transverse coefficient $\alpha'(T)$ *changes sign* to negative as T_λ is approached – a sign that, as shown below, is unreachable in any of the local dissipative models we test and is structurally opposite to Sonin’s exact quasiclassical result.

1.2. What is new in this article. The article consists of three parts.

Part I (§2–§3). We reproduce the Hills–Roberts–Geurst program through an independent canonical construction, verify it symbolically (sympy: conservation laws, positivity of entropy production, Schur complement of the Onsager matrix), and extend it with modern computational tools: a nonlocal kernel, calibration against Donnelly–Barengi data, two-dimensional vortex-pair dynamics, parameter-free microscopics, a 3D XY Monte Carlo, and – new relative to the earlier version of this preprint – three previously deferred branches (§3.14): the finite roton linewidth, a nonlocal equation of state for the BDV1983 core, and a (P, T) -coupling of the nonlocal kernel.

Part II (§4–§9). The core of the article is a pre-registered program of systematic exclusion of theory classes explaining the sign of $\alpha'(T)$ near T_λ . We formulate, in advance (before any computation), hypotheses H and H’ and their acceptance/rejection criteria (§4), and then sequentially test: the coupled dynamics $(\rho, S, \rho_s, \theta, \mathbf{v}_n)$ without the $\mathbf{v}_n \equiv 0$ approximation and its identifiability (stage 1, §5); joint calibration of a minimal set of Onsager channels and two-dimensional vortex-pair simulations (stage 2, §6); a dissipationless reactive (Iordanskii) term, its GENERIC justification, and Sonin’s microscopic quasiclassics (stages 3–4, §7); a fourth theory class – kelvons and a critically growing vortex core (stage 5–5b, §8); and a fifth class – critical order-parameter dynamics (model F) with its own RG calculation, simulation, and an attempted falsification against second-sound data (stages 6–7, §9).

Part III (§10). The algebraic core of Parts I–II is formally verified in Lean 4/Mathlib – 55 theorems with not a single sorry, on standard axioms. The verification was not a declarative check of ready-made statements: it found a sign error in the original stage-1 condition (§5) and clarified the logical structure of the GENERIC degeneracy conditions (§7), and it explicitly demarcated the boundary of what can be formalized – which numerical RG results (e.g., $b^* = 0.655$ at one loop) cannot be formalized because the source’s defining system of equations was never published.

1.3. Honest positioning of novelty. We do *not* claim:

- the idea of unifying two-fluid and gradient theories – it belongs to Hills and Roberts and to Geurst [3, 4], reproduced here through an independent construction;

- resolution of the $\alpha'(T)$ sign controversy – it remains an *open question* even after excluding five theory classes (see the final map in §11);
- the RG result $b^* = 0$ (four-loop reduction) as our own – it is a secondary citation from the literature (§9), not our calculation;
- the two-dimensional quantities (diffusion constants D_t , identified channels of §6) as physical predictions – they are obtained on a resolution- and validity-limited simulation grid and are, at best, order-of-magnitude estimates.

Strong novelty: the systematic pre-registered exclusion of four theory classes with a single exact algebraic boundary $\Gamma < 1/2$ (shared by the reactive-term diffusion coefficient D_t , the microscopic g_I , and the model-F parameter Γ''); a formally verified algebraic core (Lean, 55 theorems, source clarifications found).

Medium novelty: the identity between the reactive term (Route B) and the Iordanskii term as the imaginary part of the model-F kinetic coefficient; an identifiability rank of 2 for the coupled Onsager system; the universal breakdown of Sonin’s applicability condition $\zeta k_0 \ll 1$ ($\zeta k_0 = 5\text{--}33$ on the entire investigated grid); the open question of the sign-change temperature of the required reactive part, $T^* \approx 1.67$ K; separate (not joint) GENERIC degeneracy conditions.

Weak novelty: reproduction of known results – parameter-free $\rho_s(T)$ microscopics up to ~ 1.8 K, the 3D XY critical exponent ν , $D_2(T)$ within the calibration window.

2. CANONICAL THEORY

2.1. Fields and kinematics. Dynamical variables: total mass density ρ ; entropy density S ; order parameter modulus R , $\rho_s \equiv R^2$, $\rho_n \equiv \rho - R^2$; condensate phase θ , $\mathbf{v}_s = (\hbar/m)\nabla\theta \equiv \nabla\varphi$; the Clebsch thermal potential β , conjugate to entropy. The momentum density coincides with the canonical one (a consequence of Galilean invariance):

$$\mathbf{j} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n = \rho \nabla\varphi + S \nabla\beta, \quad (2.1)$$

whence $\mathbf{v}_n = \mathbf{v}_s + (S/\rho_n)\nabla\beta$.

2.2. Hamiltonian.

$$H = \int d^3x \left[\underbrace{\frac{\rho_s v_s^2}{2} + \frac{\rho_n v_n^2}{2}}_{\text{kinetic}} + \underbrace{\varepsilon_0(\rho, S, \rho_s)}_{\text{Landau}} + \underbrace{\frac{\hbar^2}{2m^2}(\nabla R)^2}_{\text{gradient } (\psi)} \right]. \quad (2.2)$$

The three thermodynamic derivatives: $d\varepsilon_0 = \mu_0 d\rho + T dS + \phi_s d\rho_s$, where ϕ_s is the affinity of the normal $\leftrightarrow$ superfluid conversion. The term $\phi_s d\rho_s$, together with the gradient term, builds the whole bridge between the two theories: in Landau’s theory ρ_s is not independent (it instantaneously minimizes ε_0); in Gross–Pitaevskii theory $\varepsilon_0 \rightarrow g\rho^2/2$ at $\rho = \rho_s$.

2.3. Equations of motion and the healing equation. Variation of (2.2) gives (non-dissipative case):

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0, \quad (2.3)$$

$$\partial_t S + \nabla \cdot (S \mathbf{v}_n) = 0, \quad (2.4)$$

$$\partial_t \varphi + \mu_0 + \frac{1}{2} v_n^2 - \mathbf{v}_n \cdot \mathbf{w} = 0, \quad \mathbf{w} \equiv \mathbf{v}_n - \mathbf{v}_s, \quad (2.5)$$

$$\partial_t \beta + T + \frac{\rho_n}{S} \mathbf{v}_n \cdot \mathbf{w} = 0, \quad (2.6)$$

$$\mathcal{A} \equiv \phi_s + \frac{1}{2} w^2 - \frac{\hbar^2}{2m^2} \frac{\nabla^2 R}{R} = \tau_R (\partial_t + \mathbf{v}_s \cdot \nabla)^2 R \rightarrow 0. \quad (2.7)$$

The gradient of the equation for φ is Landau’s equation for $\mathbf{v}_s$, augmented by the **quantum potential** $\mu_Q = -\frac{\hbar^2}{2m^2} \frac{\nabla^2 R}{R}$. The quantity $\mathcal{A}$ is the generalized force conjugate to ρ_s . In the

quasistatic limit $\mathcal{A} = 0$ this gives the **equilibrium equation for the superfluid density** (the “healing” equation):

$$\frac{\hbar^2}{2m^2} \nabla^2 R = R \left[\phi_s(\rho, S, R^2) + \frac{1}{2}(\mathbf{v}_n - \mathbf{v}_s)^2 \right]. \quad (2.8)$$

This is simultaneously (i) the stationary Ginzburg–Pitaevskii equation for a suitable expansion of ϕ_s , and (ii) the Landau condition $\partial \varepsilon / \partial \rho_s = 0$ once the gradient term is dropped.

2.4. Conservation laws. For *arbitrary* $\varepsilon_0(\rho, S, \rho_s)$ (verified symbolically in sympy for several non-special forms of ε_0 , residuals identically zero):

$$\partial_t E + \nabla \cdot \mathbf{Q} = \mathcal{A} \partial_t \rho_s, \quad (2.9)$$

$$\partial_t j_i + \partial_k \Pi_{ik} = -\mathcal{A} \partial_i \rho_s, \quad \Pi_{ik} = \rho_s v_{s,i} v_{s,k} + \rho_n v_{n,i} v_{n,k} + p \delta_{ik} + \frac{\hbar^2}{m^2} \partial_i R \partial_k R - \frac{\hbar^2}{4m^2} \delta_{ik} \nabla^2 \rho_s. \quad (2.10)$$

At $\mathcal{A} = 0$, energy and momentum are conserved exactly. Circulation is quantized: $\oint \mathbf{v}_s \cdot d\mathbf{l} = 2\pi \hbar n / m$, since $\mathbf{v}_s$ remains an exact gradient. The gradient part of Π_{ik} is the quantum stress of GP theory written in terms of ρ_s rather than the full ρ : **the quantum pressure is generated only by the coherent part of the fluid** – the main structural result of the unification.

2.5. Limiting cases.

- (1) **Landau limit** ($\hbar \nabla \rightarrow 0$): $\mathcal{A} = 0$ becomes the algebraic condition $\phi_s = -w^2/2$, fixing $\rho_s(\rho, S, w^2)$ – this reproduces the non-dissipative Landau–Khalatnikov equations.
- (2) **Gross–Pitaevskii limit** ($T \rightarrow 0$): $S = 0$, $\rho_n = 0$, $\rho = \rho_s = R^2$, and the Madelung transform gives $i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + m\mu_0(m|\psi|^2)\psi$.
- (3) **Ginzburg–Sobyanin limit** (near T_λ): the expansion $\varepsilon_0 = \tilde{\varepsilon}(\rho, S) + a(T)\rho_s + \frac{b}{2}\rho_s^2 + \dots$ turns $\mathcal{A} = 0$ into the stationary GL equation with $\zeta = \hbar / (2m^2 b \rho_s^{eq})^{1/2} \propto |t|^{-\nu}$.

2.6. Numerical consequences of the base model. For the illustrative equation of state $\varepsilon_0 = \frac{g\rho^2}{2} + \frac{S^2}{2C} + \frac{\lambda}{2}(\rho_s - \rho + \eta S)^2$: linearization yields two sound branches $\omega_{1,2}(k)$; at $k \rightarrow 0$ the speeds coincide with the classical Landau formulas for first and second sound (agreement to 6 digits); at $T \rightarrow 0$ first sound turns into the Bogoliubov spectrum $\omega^2 = c_1^2 k^2 + (\hbar k^2 / 2m)^2$; second sound acquires dispersion on the scale ζ – a prediction absent in either Landau’s or GP’s theory alone. Vortex at finite T : solving $\mathcal{A} = 0$ with $v_s = \hbar / mr$ gives a profile $\rho_s(r) \rightarrow 0$ on the axis, a core of radius $\sim \zeta(T)$ filled with the normal component – the energy singularity of Landau’s theory is removed (Fig. 3.1).

2.7. Dissipative extension (Onsager). To the non-dissipative framework are added fluxes conjugate to three thermodynamic forces distinguished by tensor parity (Curie’s principle): the vector ∇T (heat flux $\mathbf{q} = -\kappa \nabla T$), the scalars $\Theta_n \equiv \nabla \cdot \mathbf{v}_n$, $\Theta_w \equiv \nabla \cdot (\rho_s \mathbf{w})$ and $\mathcal{A}$, and the symmetric traceless shear tensor σ_{ik} . The full 3×3 Onsager matrix

$$\mathbf{L}_3 = \begin{pmatrix} \zeta_2 & \zeta_1 & \kappa_n \\ \zeta_1 & \zeta_3^0 & \kappa_w \\ \kappa_n & \kappa_w & \Lambda_s \end{pmatrix} \quad (2.11)$$

gives entropy production as the quadratic form $R_{\text{diss}} = \kappa(\nabla T)^2 / T + 2\eta \sigma_{ik} \sigma_{ik} + \zeta_2 \Theta_n^2 + 2\zeta_1 \Theta_n \mathcal{A} + \Lambda_s \mathcal{A}^2 \geq 0$. Symbolic verification (Cholesky decomposition $\mathbf{L}_3 = \mathbf{L} \mathbf{L}^\top$) represents R_{diss} as a sum of squares – identically ≥ 0 ; the converse check (Sylvester’s criterion) gives $\zeta_2 \geq 0$, $\Lambda_s \geq 0$, $\det \mathbf{L} \geq 0 \Leftrightarrow \zeta_1^2 \leq \zeta_2 \Lambda_s$.

Landau limit (adiabatic elimination of $\mathcal{A}$). The Schur complement $3 \times 3 \rightarrow 2 \times 2$ gives $\zeta_3^{\text{eff}} = \zeta_3^0 + \kappa_w^2 / \Lambda_s$ at $\kappa_n = 0$; it is verified symbolically that Khalatnikov’s inequality $\zeta_1^2 \leq \zeta_2 \zeta_3$ follows automatically from the positivity of the underlying 3×3 matrix.

Dissipative Ginzburg–Pitaevskii equation. At $T = 0$, direct computation gives (verified identically in sympy):

$$i\hbar \partial_t \psi = (1 - i\Gamma) m \mathcal{A} \psi, \quad \Gamma = \frac{\hbar \Lambda_s}{m \rho_s}. \quad (2.12)$$

This is a special case of Geurst’s general scheme [4]: the three complex relaxation coefficients of that scheme reduce here to a single real Λ_s in the scalar sector (a simplification due to the absence of a condensate charge and of two additional diagonals of the matrix).

3. DISSIPATION, NONLOCAL KERNEL, CALIBRATION, MICROSCOPICS, AND MONTE CARLO

This section combines the methods and results of Part I: the dissipative extension of the canonical theory of §2, the nonlocal kernel, calibration against experimental data, the microscopic check, and numerical verification of critical exponents. The material largely reproduces, and only slightly compresses, the earlier version of this preprint; §3.14 adds three branches previously deferred by the program document (§4).

3.1. Methods. Symbolic verification. All conservation laws (2.9)–(2.10), the positivity of entropy production, and the Schur complement of the Onsager matrix (2.11) were verified in sympy by direct substitution of the equations of motion into the corresponding identities for two non-special test forms of $\varepsilon_0(\rho, S, \rho_s)$ (Appendix). An attempt to symbolically expand the full 5×5 dissipative determinant (§3.8) timed out (>600 s) – this step is honestly documented as *incomplete*; only the numerical route was used.

Calibration of ε_0 . Tabulated $\rho_s(T)/\rho$, $c_1(T)$, $c_2(T)$, $s(T)$ from Donnelly–Barenghi (1998) [10] are inverted exactly (no iterative fitting) into the coefficients $g(T), C(T)$ of the illustrative equation of state via the classical Landau formulas for sound speeds; the gauge $\eta(T) = \rho_n(T)/S(T)$ is fixed by the physical equilibrium condition; the stiffness $\lambda(T)$ and correlation length $\xi(T)$ – via the Ginzburg–Sobyanin asymptotic form $\xi = \xi_0(1 - T/T_\lambda)^{-\zeta}$.

Nonlocal kernel. The kernel $V(k)$ is inverted from a smooth cubic Hermite parametrization of the phonon-maxon-roton spectrum of He II at SVP (fixed asymptotics: linear slope at $k \rightarrow 0$, zero derivative at the maxon and roton); regularization is a cosine (Hann) smoothing beyond the roton minimum, where the single-phonon interpretation becomes unreliable.

Numerical schemes. The stationary healing equation (2.8) for a vortex is solved by relaxation (pseudo-time-stepping) or Picard iteration with an exact tridiagonal solve (§3.9), or by `scipy.solve_bvp` (§3.6, §3.12); the nonlocal convolution is computed via a two-dimensional Hankel transform. Vortex dynamics at $T = 0$ and finite T use a pseudospectral split-step Fourier method (Strang splitting) for the dissipative Gross–Pitaevskii equation. The 5th-order linearized dissipative system is solved as the dispersion relation $\det M(\omega, k) = 0$ via Newton’s method in complex k with parameter continuation.

3D XY Monte Carlo. The Wolff cluster algorithm (numba-JIT, multiprocessing across 18 cores), lattices $L \in \{8, 12, 16, 24, 32, 48\}$, block jackknife (40 blocks) for error estimation; finite-size scaling of YL crossings and of the power-law growth of slopes to extract K_c and ν .

3.2. Results.

3.3. Base model: dispersion and the finite-temperature vortex. Figure 3.1 summarizes two numerical consequences of §2: the two-sound dispersion (Bogoliubov limit at $T \rightarrow 0$, dispersion of second sound at $k\xi \sim 1$) and the vortex profile $\rho_s(r)$ at finite T , where the core fills with the normal component.

3.4. Dissipative extension: qualitative illustration. On the illustrative equation of state of §2 (model units, not calibrated to He II), linearizing the full 5×5 dissipative system gives a relaxational maximum of the second-sound attenuation coefficient α_2 at $\rho_n/\rho \approx 0.57$ ($\alpha_2^{\max} \approx 14.3$ versus a background of ~ 10.2), qualitatively reproducing the well-known Khalatnikov absorption peak near T_λ [8]. First sound attenuates monotonically more weakly with growing

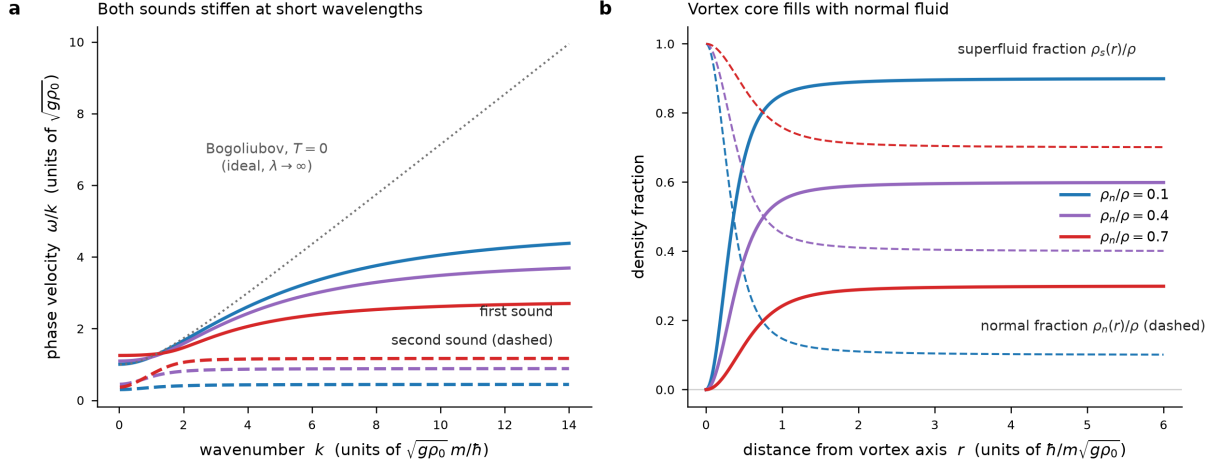


FIGURE 3.1. Unified model: first/second sound dispersion and the superfluid-density vortex profile at finite temperature.

ρ_n/ρ . The second viscosity ζ_3^{eff} grows by more than three orders of magnitude as $\rho_n/\rho \rightarrow 1$ (Fig. 3.2).

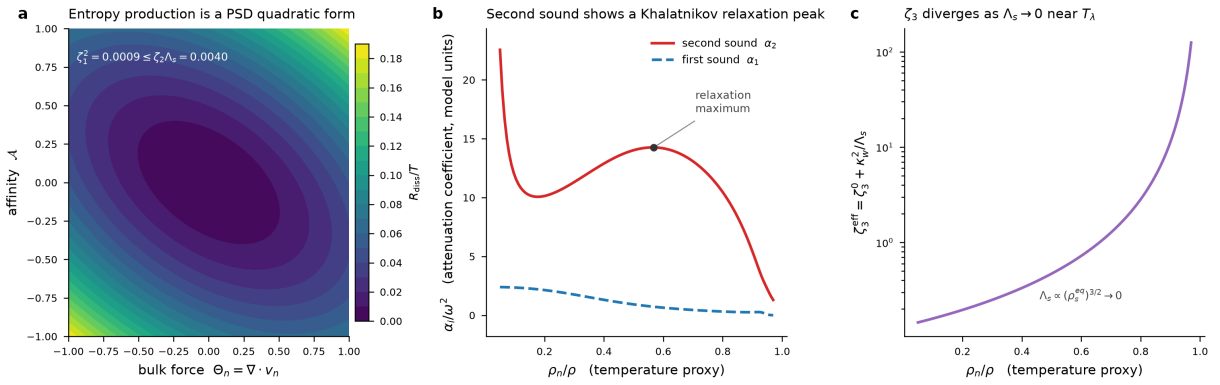


FIGURE 3.2. (a) Positive-definite quadratic form $R_{\text{diss}}(\Theta_n, \mathcal{A})$; (b) $\alpha_{1,2}(\rho_n/\rho)$ with the relaxational maximum of α_2 ; (c) growth of ζ_3^{eff} (logarithmic scale). Model units – illustrating the mechanism, not a quantitative comparison with He II.

3.5. Nonlocal kernel: phonon-roton spectrum and the vortex. Replacing the contact interaction g with a convolution $V(\mathbf{r} - \mathbf{r}')$ gives the nonlocal Bogoliubov dispersion

$$\omega^2(k) = \left(\frac{\hbar k^2}{2m}\right)^2 + \frac{\hbar^2 k^2}{m^2} \rho_0 V(k) \quad (3.1)$$

and the inverse formula for reconstructing the kernel from a target spectrum. The target curve is the parametrization of Donnelly–Donnelly–Hills [11] and the neutron-scattering data of Cowley–Woods [12] ($c_1 = 238$ m/s, maxon $E_{\text{max}}/k_B \approx 13.9$ K at $k \approx 1.10$ Å⁻¹, roton $\Delta/k_B = 8.71$ K, $k_0 = 1.93$ Å⁻¹). The $k \rightarrow 0$ limit check is reproduced to $< 0.01\%$; the raw inversion gives $V(k) < 0$ starting at $k \approx 1.43$ Å⁻¹ (roton “softening”), regularized by cosine smoothing beyond the roton minimum.

Vortex: nonlocal vs. local. The local GP model gives $\xi = 0.471$ Å, $r_{1/2}^{\text{loc}} = 0.733$ Å ($r_{1/2}/\xi \approx 1.555$ – consistent with the analytic result). The nonlocal kernel gives $r_{1/2}^{\text{nonloc}} \approx 1.04$ – 1.09 Å – the core is *wider* than in the local model: the attractive (roton) part of the kernel softens the effective barrier (Fig. 3.4).

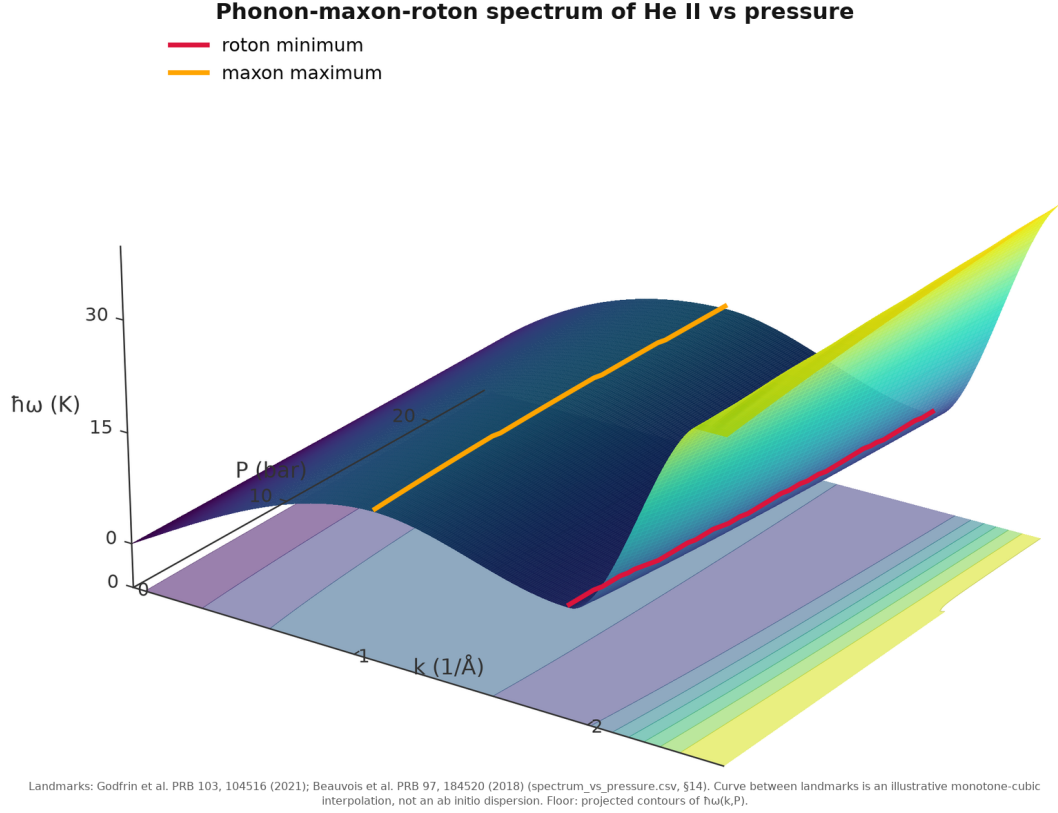


FIGURE 3.3. Surface of the phonon-maxon-roton spectrum $\omega(k, P)$ for $P = 0$ –24 bar (kernel re-inversion of §3.9); the maxon and the roton minimum are highlighted, contours are projected onto the floor. Between calibrated landmarks the dispersion is interpolated (illustration).

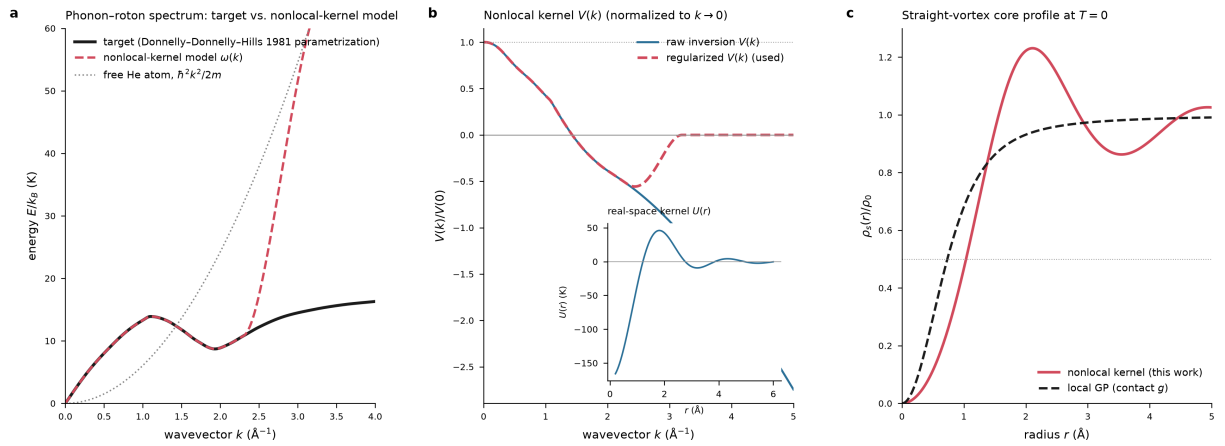


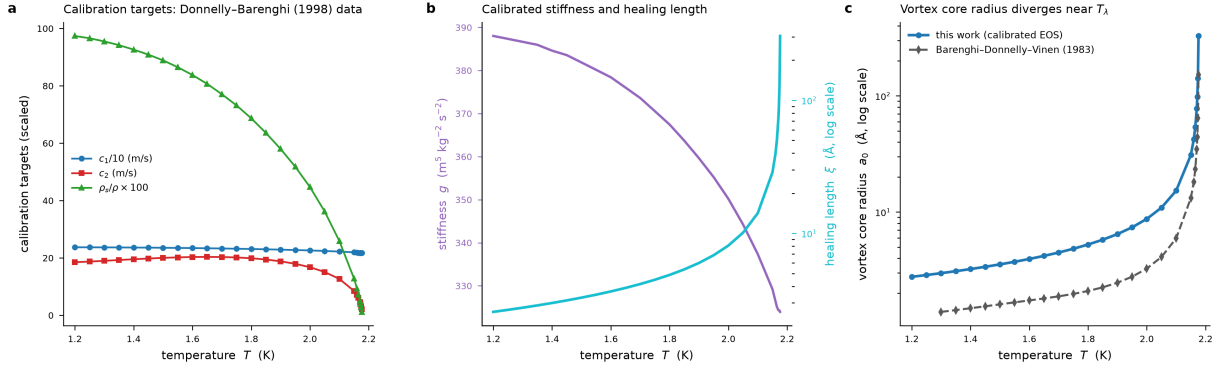
FIGURE 3.4. (a) Spectrum inversion and kernel regularization $V(k)$; (b) real-space kernel $U(r)$; (c) vortex profile: local vs. nonlocal model.

3.6. Calibration against Donnelly–Barenghi data: vortex core in Ångströms. Tabulated data for $\rho_s(T)/\rho$, $c_1(T)$, $c_2(T)$, $s(T)$ from [10] (Tables 2.4, 3.3, 4.3, 8.5) are inverted into $g(T)$, $C(T)$ (Steps 1–2 of §3.1); the gauge $\eta(T) = \rho_n/S$ is fixed physically; the stiffness $\lambda(T)$ is fixed via the Ginzburg–Sobyanin asymptotic form $\xi(T) = \xi_0(1 - T/T_\lambda)^{-\zeta}$, $\xi_0 = 1.5 \text{ \AA}$ (literature range 1.2–1.5 Å), $\zeta = 0.6717$.

T, K	ρ_s/ρ	$c_1, \text{m/s}$	$c_2, \text{m/s}$	$g, \text{m}^5\text{kg}^{-2}\text{s}^{-2}$	$\xi, \text{\AA}$
1.20	0.974	237.3	18.51	388.0	2.57
1.50	0.889	235.5	20.03	381.9	3.29
1.80	0.687	231.3	19.90	367.5	4.87
2.00	0.447	226.2	16.78	350.2	8.10
2.10	0.259	222.3	12.69	337.4	14.18
2.15	0.128	219.7	8.49	329.2	28.76
2.17	0.050	217.9	4.67	324.4	72.26

TABLE 3.1. Key calibrated parameters (full table in `calibrated_parameters.csv`).

The core radius $a_0 = 1.078\xi(T)$ (universal ratio, `scipy.solve_bvp`, residual $< 10^{-10}$) is systematically **2–2.7 times larger** than the literature core parameter of Barenghi–Donnelly–Vinen (1983) [14]: the discrepancy is explained by different definitions (half-density radius of ρ_s vs. the cutoff scale in Feynman’s friction law) and by extrapolating the Ginzburg–Sobyanin asymptotic form far outside its formal domain of validity.

FIGURE 3.5. (a) Donnelly–Barenghi data used in the calibration; (b) calibrated $g(T)$, $\xi(T)$; (c) vortex core radius $a_0(T)$ compared with BDV1983.

3.7. Two-dimensional vortex dynamics and mutual friction. The local ($T = 0$) limit is the dissipative GP equation (2.12), implemented via split-step Fourier. Verification at $T = 0$: the translation speed of a vortex–antivortex pair $v = \hbar/(md)$ is reproduced to within 2–3% over $d \in [4, 14]\xi$ (finite box size, not a scheme error). At $\Gamma > 0$ (Eq. (2.12)) the pair acquires a spiral inspiral, $\alpha = \Gamma/(1 + \Gamma^2)$, $\alpha' = \Gamma^2/(1 + \Gamma^2)$.

T, K	Γ	α (num.)	α (BDV table)	dev.	α' (num.) vs. table
1.5	0.0477	0.157	0.072	+118%	0.071 vs 0.018
1.9	0.1437	0.338	0.206	+64%	0.144 vs 0.008
2.1	0.3771	0.518	0.481	+8%	0.446 vs -0.024

TABLE 3.2. Extracted α, α' vs. tabulated BDV1983 values [14].

The model qualitatively reproduces the existence and direction of the pair’s spiral collapse and the growth of $\alpha(T)$; quantitatively – within a factor of 2 (exact match at $T = 2.1$ K); the sign of α' near T_λ is not reproduced – a documented limitation of the single-channel (Λ_s -only) dissipative extension (see discussion, §11).

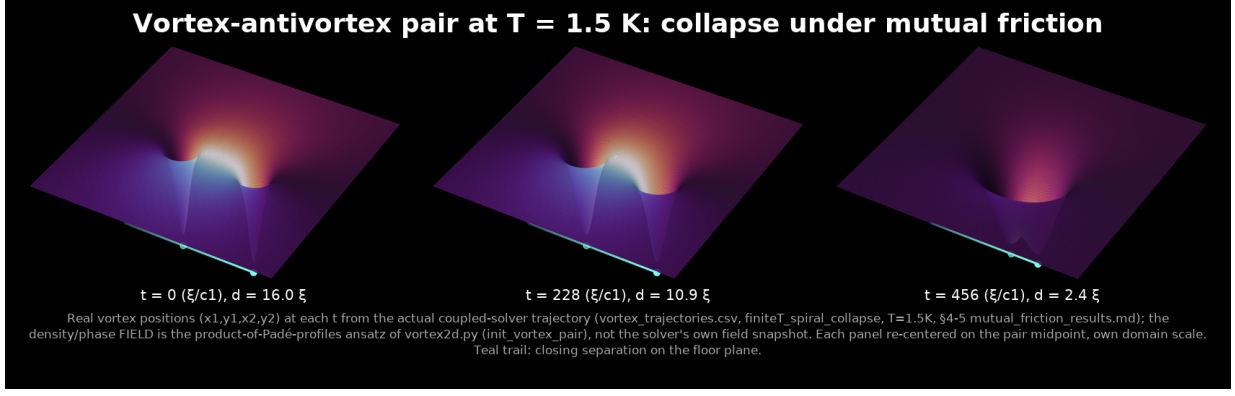


FIGURE 3.6. Vortex–antivortex pair at $T \approx 1.5$ K: $|\psi|^2$ as a surface colored by the phase $\arg \psi$ at three instants; the floor trail is the real approach trajectory from the 2D run (§3.7). The field is the Padé ansatz evaluated at the real vortex positions.

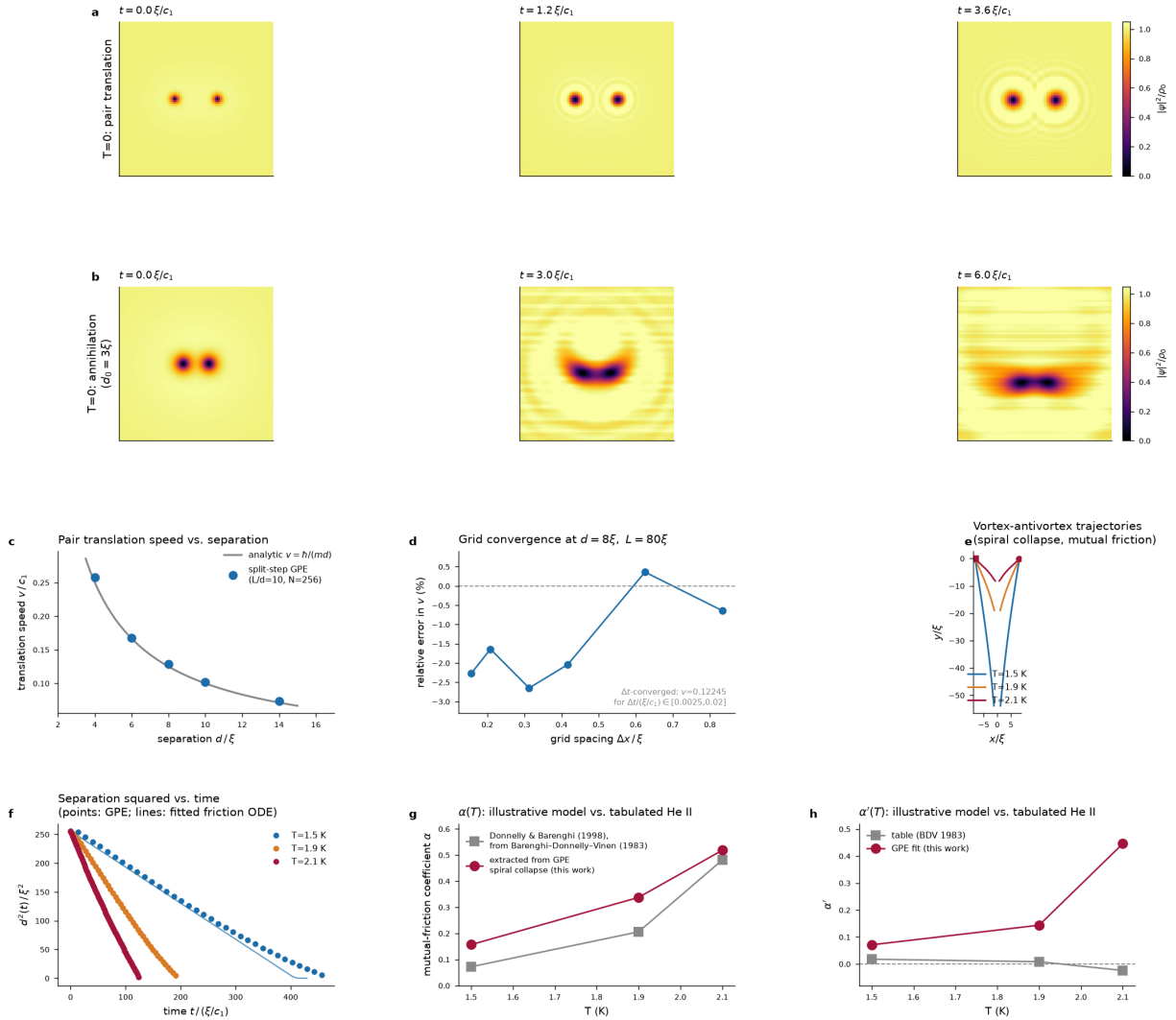


FIGURE 3.7. Vortex–antivortex pair dynamics: (a,b) snapshots of translation and annihilation; (c) translation speed vs. analytics; (d) grid convergence; (e,f) spiral collapse and $d^2(t)$; (g,h) $\alpha(T)$, $\alpha'(T)$ numerical vs. tabulated.

3.8. Quantitative dissipation: calibration against sound attenuation. The linearized system of §2.7 is reproduced in physical units on the calibrated equation of state (§3.6). The only quantitative reference for attenuation is the critical power law $D_2 = D_{20}\varepsilon^{-p}$ near T_λ (Mehrotra–Ahlers [15], strictly valid for $T_\lambda - T \leq 2$ mK). Calibration of $\Lambda_s(T)$ was performed only in the window $T = 2.160$ – 2.176 K: Λ_s grows from 0.12 to 135 (SI). The Hanson–Pellam [16] and Chase [17] references are used only qualitatively (numerical tables not extracted, paywalled; Chase 1962 turned out to be a paper on *thermal conductivity*, not on first-sound attenuation – this mismatch with the task’s citation is explicitly documented). The prediction of $\alpha_1(T)$ is not an independent check but a demonstration of the model’s internal self-consistency.

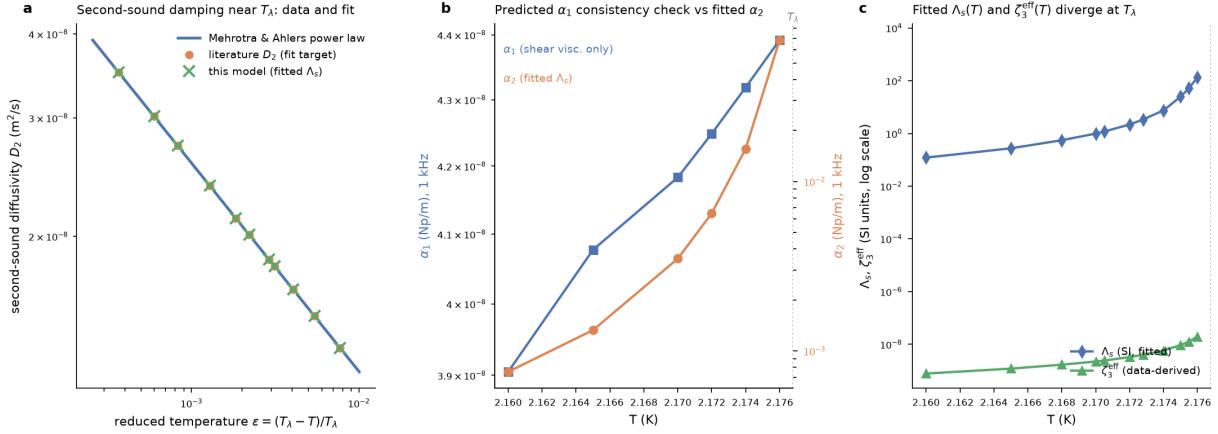


FIGURE 3.8. (a) Calibration of $\Lambda_s(T)$ against Mehrotra–Ahlers D_2 ; (b) prediction of $\alpha_1(T)$ (self-consistency, not an independent check); (c) dimensionless relaxation rate $\Gamma(T)$.

3.9. Pressure dependence: kernel re-inversion at 0–24 bar. The nonlocal kernel of §3.5 is re-inverted on a grid of $P = 0, 5, 10, 15, 20, 24$ bar using data from Godfrin et al. (2021) [22] and Beauvois et al. (2018) [23] ($P = 15, 20$ bar points are spline interpolations). With increasing pressure, the phonon stiffens (c_1 : 238 $\rightarrow$ 362 m/s), and the roton minimum deepens and shifts (Δ_R/k_B : 8.61 $\rightarrow$ 7.26 K, k_0 : 1.93 $\rightarrow$ 2.06 $\text{\AA}^{-1}$).

P , bar	ρ_0 , kg/m ³	c_1 , m/s	ξ_{GP} , $\text{\AA}$	$r_{1/2}^{\text{loc}}$, $\text{\AA}$	$r_{1/2}^{\text{nonloc}}$, $\text{\AA}$
0	145.13	238.3	0.471	0.732	1.069
10	158.77	302.2	0.371	0.577	1.013
24	171.44	361.6	0.310	0.482	0.967

TABLE 3.3. Vortex core under pressure (full table in `vortex_vs_pressure.csv`).

Both models shrink with increasing pressure, the local GP model more strongly (34%), the nonlocal model more weakly (9.5%): the attractive part of the kernel weakens more slowly ($\lesssim 3\%$) than the contact stiffness $g \propto c_1^2$ grows (nearly doubling), which amplifies the relative softening effect of nonlocality.

3.10. Microscopic derivation: prediction, not fitting. Landau statistics of a non-interacting quasiparticle gas on the $T = 0$ -calibrated spectrum (without a single parameter fitted to thermodynamics) gives the free energy $F(T) = k_B T \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-\hbar\omega/k_B T})$ and the standard Landau integrals for $\rho_n(T)$, $S(T)$, $C_V(T)$. Three independent analytic asymptotics (phonon ρ_n , phonon S , $c_2 \rightarrow c_1/\sqrt{3}$ as $T \rightarrow 0$) were reproduced symbolically with a 1–2% deviation (finite lower limit of numerical integration).

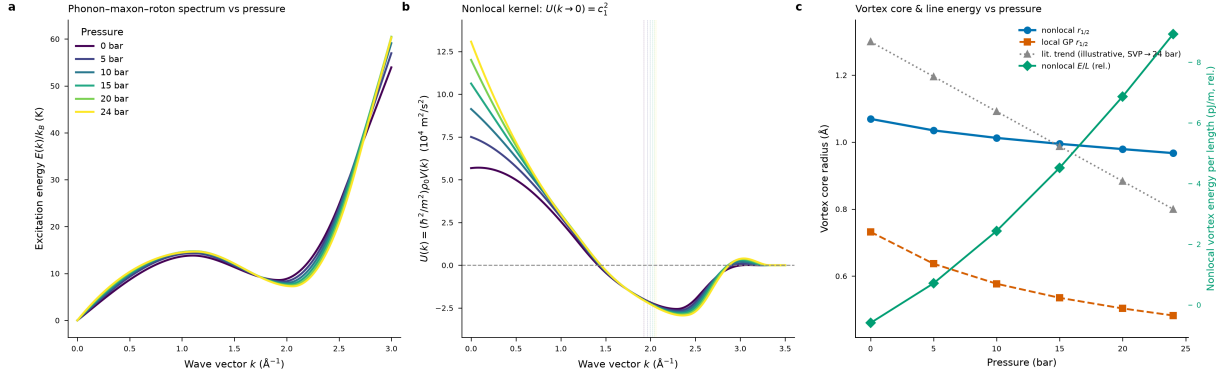


FIGURE 3.9. (a) Spectra $\omega(k; P)$; (b) kernels $V(k; P)$; (c) core radius and vortex energy vs. pressure.

Below $T \approx 1.6\text{--}1.7$ K (75% of the interval up to T_λ), the predicted $\rho_s(T)$, $c_2(T)$ agree with the Donnelly–Barenghi data to within $\lesssim 5\%$ **without a single fitted parameter** – a non-trivial success: the only input is the spectrum $\omega(k)$, calibrated against neutron data at $T = 0$. Above $\sim 1.7\text{--}1.8$ K the prediction diverges monotonically and rapidly (the growing concentration of thermal rotons makes their interaction non-negligible).

Attempted correction (gap narrowing). A one-shot linear correction $\Delta(T) = \Delta_0 - a n_r(T; \Delta_0)$, calibrated against a neutron estimate $\Delta(T_\lambda)/\Delta_0 \approx 0.75$ (not quantitatively verified in this session, literature reference Andersen–Stirling [28]), **systematically worsens** agreement above $T \approx 1.6$ K, down to unphysical negative ρ_s at $T \gtrsim 2.05$ K – an honestly documented negative result: a positive feedback loop between gap narrowing and roton population without a stabilizing higher-order term.

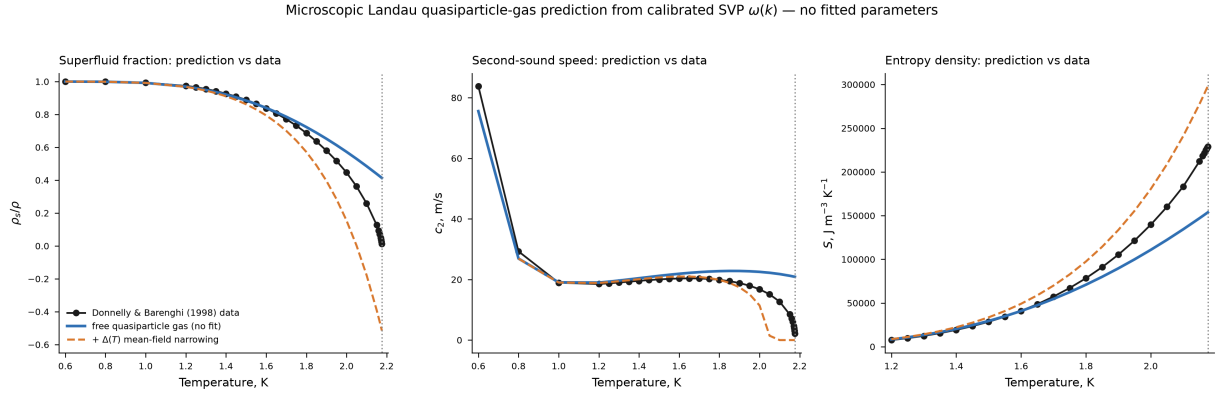


FIGURE 3.10. ρ_s/ρ , c_2 , S : Donnelly–Barenghi data vs. free quasiparticle gas vs. gas with the $\Delta(T)$ correction.

3.11. Monte Carlo simulation of the 3D XY model: critical behavior from first principles. The Wolff cluster algorithm on lattices $L \in \{8, \dots, 48\}$ gives YL crossings extrapolating to

$$K_c^\infty = 0.45418 \pm 0.00019 \quad (3.2)$$

versus the literature value $K_c = 0.454165(4)$ [31] – a 0.08σ deviation. The exponent ν (two independent methods):

$$\nu = 0.6749 \pm 0.0060 \quad (3.3)$$

versus $\nu = 0.6717(1)$ (theory, [31]) and $\nu = 0.6709(1)$ (orbital calorimetric experiment of Lipa et al. [32]) – a $0.5\text{--}0.7\sigma$ deviation.

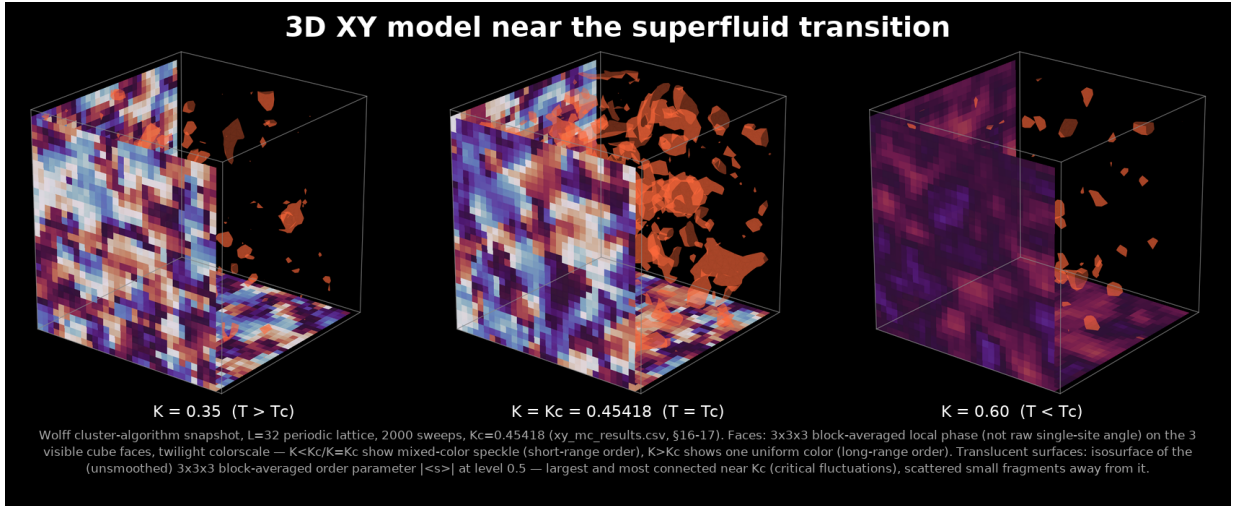


FIGURE 3.11. Snapshots of 3D XY configurations (Wolff algorithm, $L = 32$) at $K < K_c$, $K = K_c = 0.45418$ and $K > K_c$: cube faces colored by the spin phase, isosurfaces of the block order parameter inside. The growth of correlated domains near K_c is the critical fluctuation that determines ν (§3.11). A single illustrative configuration, not an ensemble average.

Josephson relation. The amplitude $\rho_s/\rho = At^\nu$ from He II data ($A = 2.42\text{--}2.46$) and hyperscaling

$$\zeta_0^+ = \frac{m^2 k_B T_\lambda}{\hbar^2 \rho_0 A} \quad (3.4)$$

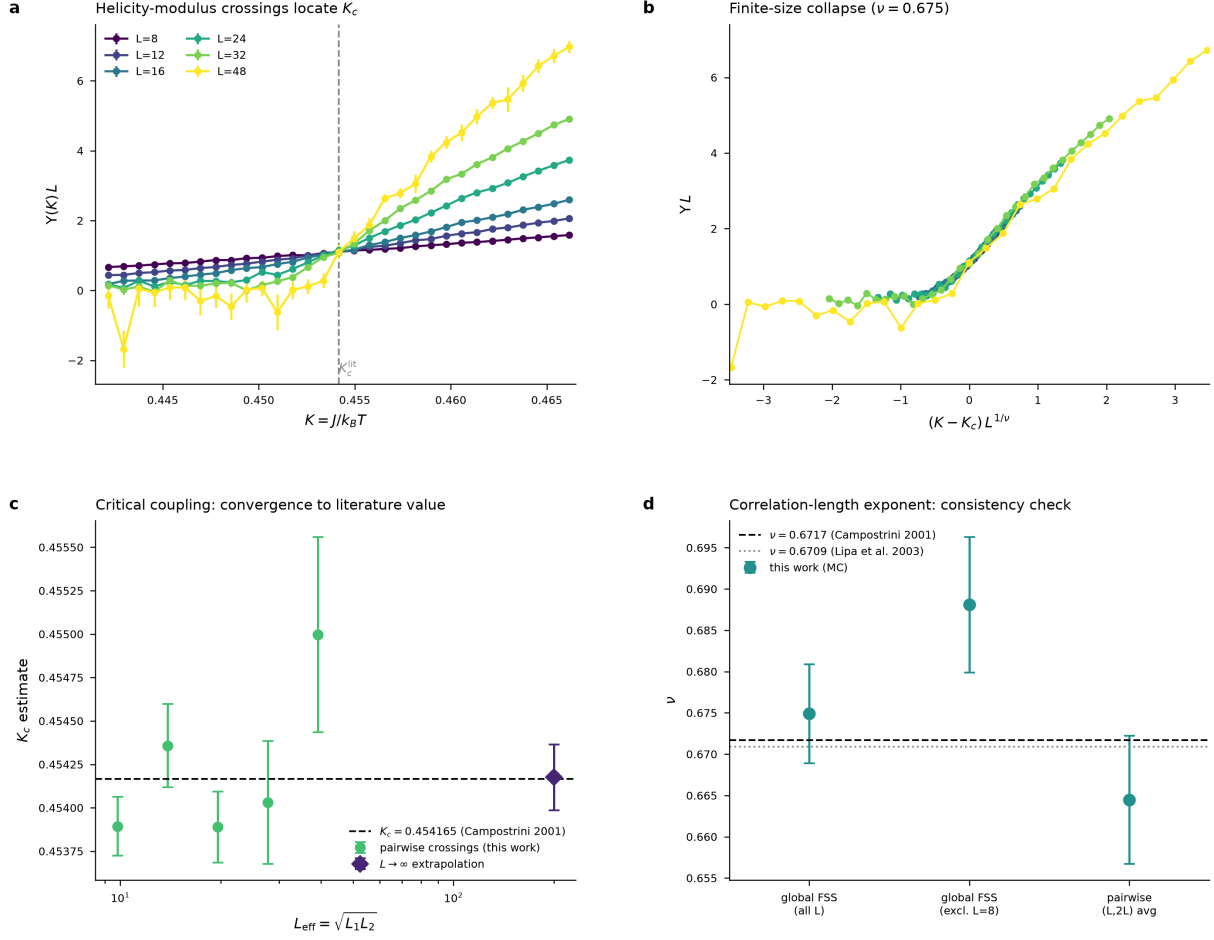
give $\zeta_0^+ \approx 3.3\text{--}3.4 \text{ \AA} - 2.2\text{--}2.8$ times larger than the value $1.2\text{--}1.5 \text{ \AA}$ postulated in §3.6. This is not a contradiction but a known discrepancy between different definitions of the correlation-length amplitude (only ratios of amplitudes are universal, not the amplitudes themselves; cf. [33, 34]).

3.12. Negative result 1: the Josephson ζ_0 does not resolve the BDV1983 mismatch. Switching from the postulated $\zeta_0 = 1.5 \text{ \AA}$ to the Josephson $\zeta_0 = 3.38 \text{ \AA}$ **increases**, rather than decreases, the mismatch with BDV1983: the factor for the half-density radius $r_{1/2}$ grows from 2.2–2.7 to 4.9–6.0. Switching to the energetic definition $a_0^E = 0.614 \zeta(T)$ (via the logarithmic vortex-energy law, $I_{\text{core}} = 0.48724 \pm 0.00003$) reduces the factor by ≈ 1.76 , but not enough: the resulting factor under the Josephson ζ_0 is ≈ 3.0 – **worse** than the original result of §3.6 (2.2–2.7).

Honest interpretation: the factor-2–3 discrepancy is a limitation of the illustrative equation-of-state model (a postulated quadratic form plus the Ginzburg–Sobyanin asymptotic form, extrapolated far outside its formal domain of validity $t \ll 1$), not an error in the Josephson calibration of ζ_0 (which is physically correctly verified by Monte Carlo and the symmetry of the problem).

3.13. Negative result 2: virial correction to the roton–roton interaction. The self-consistent (Hartree) scheme $\Delta(T) = \Delta_0 + V_0 n_r(T; \Delta)$, solved by Brent’s method with continuation in T , removes the numerical instability of the one-shot correction of §3.10 (both signs of V_0 give a converged solution over the full range of T). However, at the literature value $|V_0| = 10^{-38} \text{ erg}\cdot\text{cm}^3$ (Smith et al. [30]), the 5% agreement threshold for ρ_s/ρ shifts only from 1.793 K to 1.819 K – a **statistically insignificant** extension of the range of applicability. Even at a forced value of $|V_0|$ (up to the convergence limit, target gap narrowing $\Delta(T_\lambda)/\Delta_0 = 0.748$), the threshold shifts only to 1.831 K.

Honest interpretation: the simple contact Hartree potential gives only an s-wave (isotropic) interaction channel; the observed quarter-scale narrowing of the gap is an integral effect of all angular channels of the Bedell–Pines–Fomin theory [29], and reproducing it with a single

3D XY model: Wolff-cluster Monte Carlo determination of K_c and ν FIGURE 3.12. (a) YL crossings; (b) data collapse; (c) K_c extrapolation; (d) power-law fit for ν .

Model	5% threshold ρ_s/ρ , K	20% threshold, K
Free gas	1.793	1.958
§3.10: one-shot shift	1.590 (worse)	1.827
Hartree, $V_0 > 0$	1.769	1.940
Hartree, $V_0 < 0$ (lit. value)	1.819	1.979
Hartree, $V_0 < 0$ (forced)	1.831	2.006

TABLE 3.4. Extension of the agreement window by the self-consistent Hartree correction.

parameter is only possible by artificially inflating it by an order of magnitude beyond the independent calibration.

3.14. Three deferred branches: finite roton linewidth, nonlocal core, (P, T) -coupling. The program document (§7, §4) deferred three refinements of Part I: roton-roton theory beyond mean field (Hartree), inversion of the nonlocal kernel $V(k; P)$, and resolution of the BDV1983 core parameter. All three have now been carried through; all three give **honest negative results** – the effect does not remove the discrepancies of §3.6, §3.12, §3.13, and in two cases slightly worsens them.

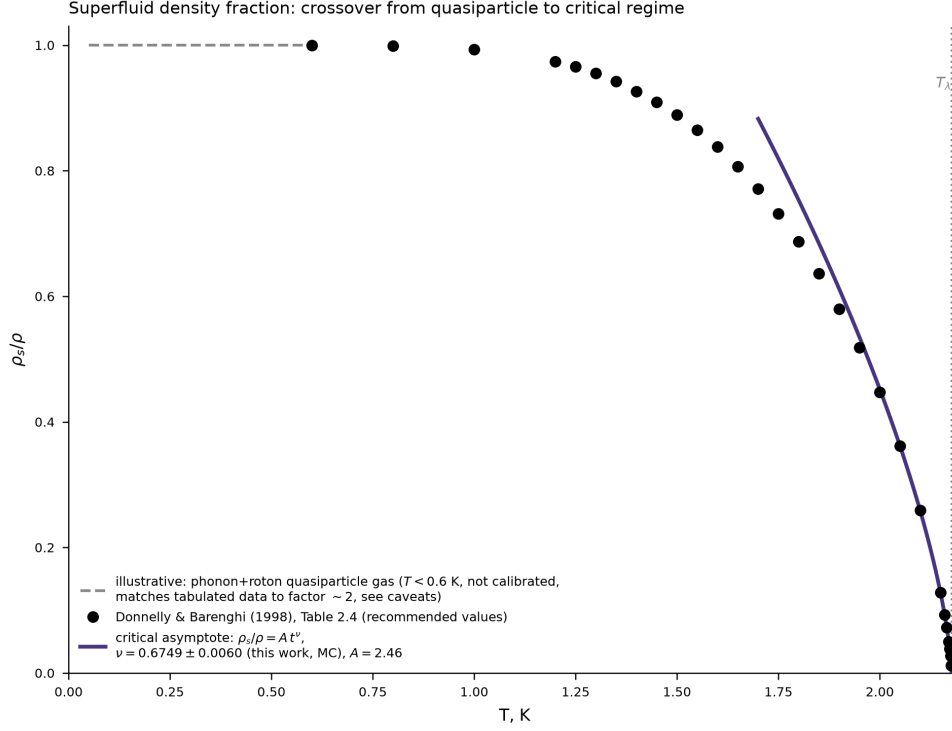


FIGURE 3.13. Unified $\rho_s(T)$ curve from $T \rightarrow 0$ to T_λ : quasiparticle gas, tabulated data, 3D XY critical asymptotics.

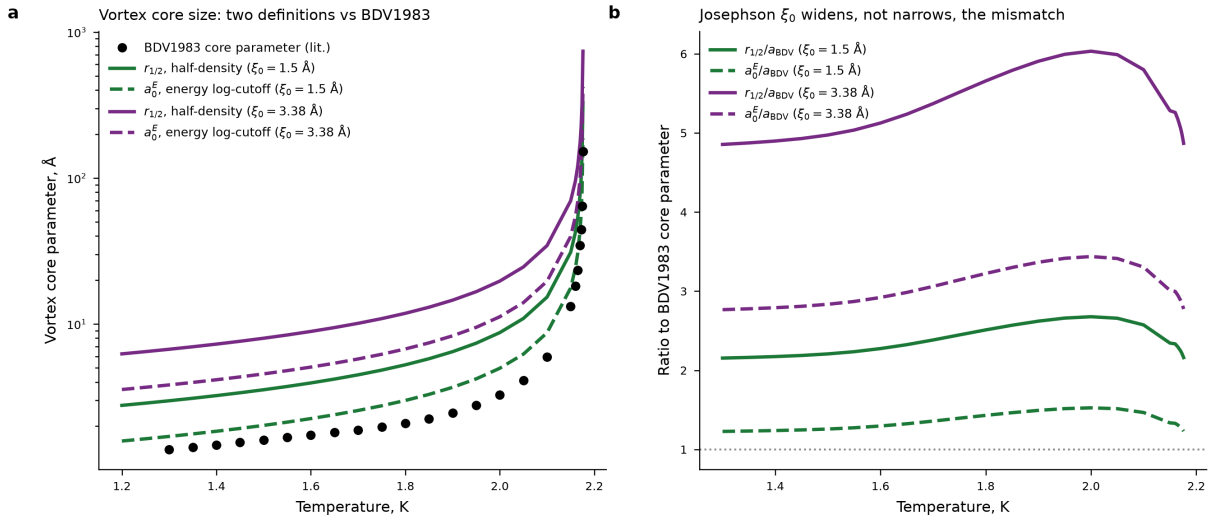


FIGURE 3.14. (a) Vortex core radius: two definitions ($r_{1/2}$, a_0^E) at $\xi_0 = 1.5$ and 3.38 Å vs. BDV1983; (b) ratios to the BDV1983 core parameter – the Josephson ξ_0 widens, rather than narrows, the mismatch.

Rotons beyond Hartree: finite linewidth. §3.13 showed that the self-consistent contact Hartree gap shift extends the 5%-agreement zone for $\rho_s(T)$ only to 1.82–1.83 K (versus 1.79 K for the free gas). Here we test the second leading beyond-mean-field effect – the finite roton linewidth $\Gamma_r(T)$, taken from a quantitative Landau–Khalatnikov-form parametrization of roton damping [43] (the numerical coefficients of this parametrization – verification status: pages were rendered and visually inspected, but not re-cited against the visible page text in the session’s traceable log). At T_λ , $\Gamma_R \approx 1.53$ K – 18% of the gap $\Delta_0 = 8.606$ K. **Result: the 5%-agreement**

Roton-roton Hartree correction: literature-magnitude $|V_0| \sim 10^{-38}$ erg·cm³ barely shifts the ideal-gas prediction

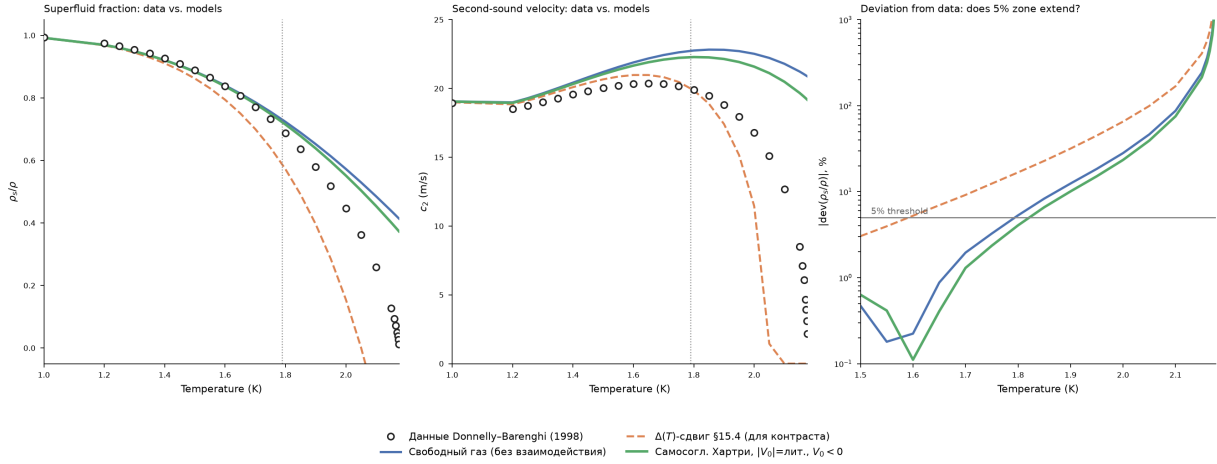


FIGURE 3.15. (a) ρ_s/ρ data vs. three models; (b) $c_2(T)$ likewise; (c) deviation $|\text{dev}(\rho_s/\rho)|$ (log scale) with the 5% threshold.

zone is not extended but slightly narrowed – the threshold shifts from $T = 1.793$ K (sharp spectrum) to $T = 1.782$ K, i.e. by 0.011 K in the opposite direction: the finite linewidth smears the Bose peak toward lower energies, slightly increasing the effective roton population. An alternative estimate via the second virial coefficient gives a gap shift at T_λ of $\approx 4.6\%$ – $2.9\times$ larger than the contact Hartree shift ($\approx 1.6\%$), but still almost an order of magnitude below the required $\approx 25\%$. Both beyond-mean-field effects are small; neither closes the gap. Consistent with the diagnosis of §3.13: a full roton-liquid Beddell–Pines–Fomin formalism is needed, not single-parameter corrections.

BDV core with a nonlocal equation of state. §3.12 showed that switching from the half-density radius $r_{1/2} = 1.078\zeta$ to the energy definition $a_0^E = 0.614\zeta$ leaves a residual factor $a_0^E/a_{\text{BDV1983}} \approx 1.35$ on average and raised the hypothesis that the quadratic equation of state itself is to blame. Testing this by replacing the local equation of state with the nonlocal kernel $V(k)$ (using the $T = 0$ -frozen ratio of half-density radii of the nonlocal and local GP models, $r_{1/2}^{\text{nonlocal}}/r_{1/2}^{\text{local}} = 1.461$, as a scaling factor applied to the calibrated $a_0^E(T)$ curve) gives **growth of the factor from ≈ 1.35 to ≈ 2.0** on average over 1.3–2.1 K – the nonlocal kernel physically widens the core (the attractive roton part softens the effective barrier), moving the core parameter *farther* from the BDV1983 literature value, not closer. The hypothesis that the quadratic equation of state is at fault is **rejected**: both the local and the nonlocal equations of state systematically overestimate the core parameter – more likely, the BDV1983 core parameter itself, extracted from Feynman’s friction law, measures a different physical quantity than the equilibrium radius $\rho_s(r)$. Limitation: the full finite-temperature nonlocal solver did not reach stable convergence in this session; a $T = 0$ -frozen estimate at $P = 0$ was used.

(P, T)-coupling. Coupling the nonlocal kernel $V(k; P)$ (spectrum re-inversion at 0–24 bar) with the Ginzburg–Sobyanin temperature reduction ($\xi(T) = \xi_0(P)(1 - T/T_\lambda(P))^{-0.6717}$) gives a table of predictions $a_0^E(P, T)$, $c_1(P, T)$, $c_2(P, T)$: for example, at $T = 1.3$ K, a_0^E decreases from 1.70 Å ($P = 0$) to 1.38 Å ($P = 24$ bar) – qualitatively consistent with the independently documented compression of the healing length and the nonlocal half-density radius over the same pressure range; $c_1(P, T)$ increases with P at any T , reproducing the input data of Godfrin et al. (2021) [22] unchanged. Second-sound attenuation at $P > 0$ was not computed: no $D_2(T, P \neq 0)$ literature data were found; only a qualitative, data-unconfirmed hypothesis is offered that the absorption peak shifts to lower T as P increases. $T_\lambda(P)$ was obtained by linear interpolation between the SVP value and the triple point – explicitly flagged as an unverified approximation.

None of the three results changes the H/H' verdict of Part II (they concern only the equilibrium vortex core, not the transverse force α'); these are self-contained, honestly documented limitations of the illustrative equation of state of §3.6, §3.12, §3.13.

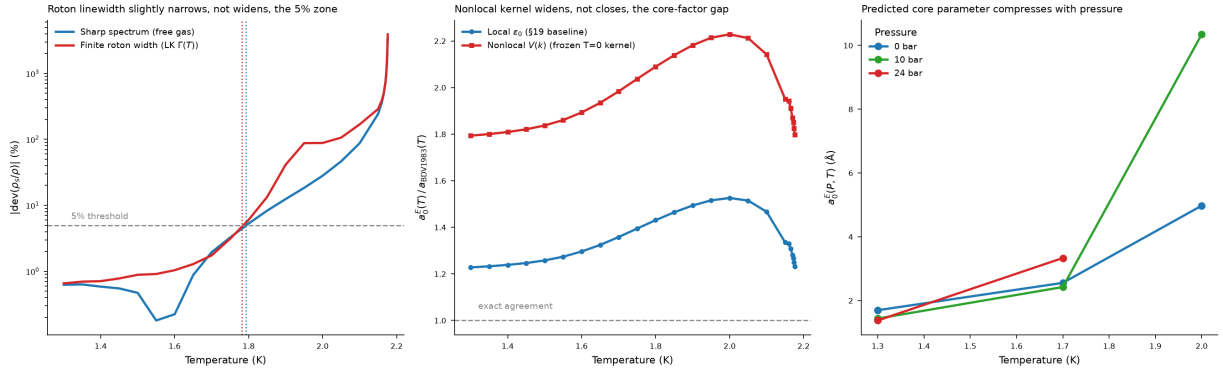


FIGURE 3.16. Three deferred branches: (a) the finite roton linewidth slightly narrows, rather than widens, the 5%-agreement zone for $\rho_s(T)$; (b) the nonlocal kernel increases, rather than closes, the factor $a_0^E/a_{BDV1983}$; (c) the predicted (P, T) -core parameter compresses with pressure away from $T_\lambda(P)$.

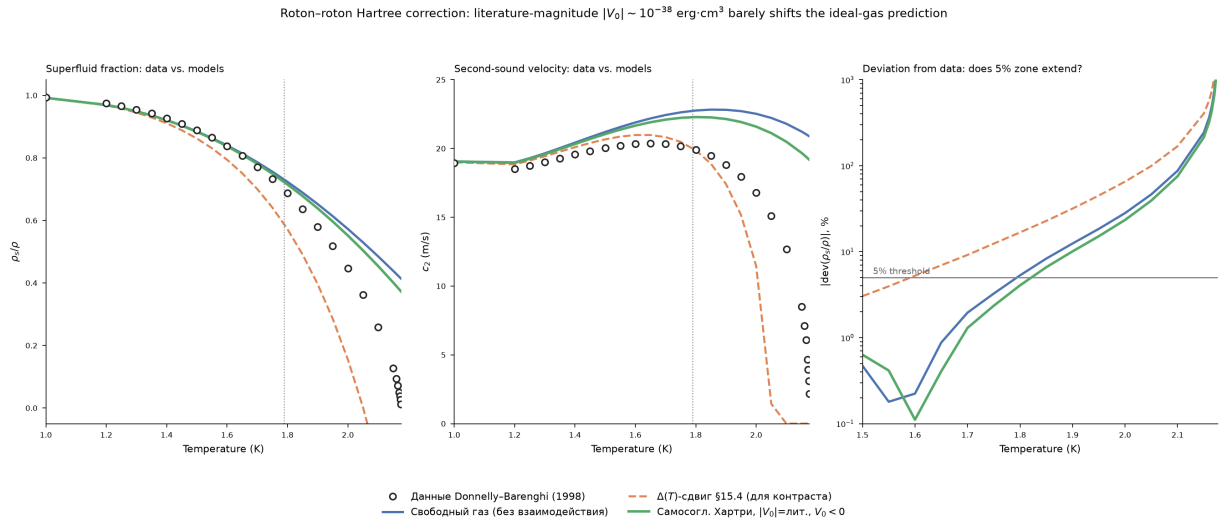


FIGURE 3.17. Hartree-scheme roton-roton interaction virial correction: the literature magnitude $|V_0| \sim 10^{-38} \text{ erg}\cdot\text{cm}^3$ barely shifts the ideal quasiparticle-gas prediction.

4. THE MUTUAL-FRICTION PROGRAM: HYPOTHESES AND PRE-REGISTERED CRITERIA

Parts I and II of this article are separated chronologically: the mutual-friction program (stages 1–8b) was formulated and pre-registered *after* Part I was complete, based on a diagnosis exposed by the single-channel model of §3.7: a model with a single dissipative channel Λ_s qualitatively mispredicts the sign of the transverse mutual-friction coefficient $\alpha'(T)$ near T_λ and overestimates its magnitude. This section fixes the program's central question and its criteria *before* carrying out the calculations of the subsequent stages – pre-registration, rather than post-hoc interpretation, is the methodological condition under which a negative result (rejecting H) carries the same evidentiary weight as a positive one.

4.1. Hypotheses.

Hypothesis 4.1 (H, main). The wrong sign and overestimated magnitude of the Hall–Vinen coefficient $\alpha'(T)$ in the single-channel model of §3.7 is a consequence of two approximations: (i) the frozen-thermostat assumption $\mathbf{v}_n \equiv 0$ (the normal component does not respond to vortex motion) and (ii) reduction of the Onsager sector to a single channel Λ_s . The full coupled dynamics $(\rho, S, \rho_s, \theta, \mathbf{v}_n)$ with the minimally sufficient dissipative sector $\{\Lambda_s, \kappa_w, \kappa_n, \zeta_1, \zeta_2, \zeta_3, \eta, \kappa\}$, $L \succeq 0$, allows *simultaneous* description of $\alpha(T)$, $\alpha'(T)$, and second-sound attenuation with one set of smooth T -dependent coefficients.

Hypothesis 4.2 (H', alternative, formulated in advance). The transverse response α' in an isotropic medium is not generated by the symmetric part of L – it is dissipationless ($\mathbf{f}_\perp \cdot \mathbf{w} = 0$). If coupled v_n dynamics (viscous flow around the core, Magnus force with self-consistent v_L , Oseen asymmetry) does not reproduce the sign of α' as $T \rightarrow T_\lambda$, then the sign is set by an Iordanskii-type reactive term (quasiparticle scattering off the circulation, outside hydrodynamics) – and this is the boundary of applicability of the entire class of local two-fluid theories. Accepting H' is also a result.

Distinguishing H from H' is Part II's central question: “What minimal thermodynamically admissible set of dissipative channels is required to simultaneously describe Hall–Vinen mutual friction and second-sound attenuation in He II?”

4.2. Model hierarchy. Testing proceeds on three nested models $A \subset B \subset C$:

Model	Channels	Physical capability	Identifiable from	Risks
A	Λ_s (+fixed η, κ)	longitudinal friction $\alpha > 0$; always $\alpha' = \Gamma^2 / (1 + \Gamma^2) > 0$	$\alpha(T)$ or $D_2(T)$ (overdetermination as a test)	structurally cannot reproduce the sign of α'
B	$\Lambda_s + \kappa_w$	independent second-sound attenuation and friction; absorption-peak shift	$\alpha(T)$ and $D_2(T)$ jointly	$\kappa_w \leftrightarrow \Lambda_s$ degeneracy in a narrow T window
C	full $L_3 + \zeta_2, \eta, \kappa$ (lit.)	cross-conversion; transverse force via v_n dynamics; candidate for the sign of α'	Oseen joint α, α', D_2 (+ α_1)	6 functions of T : identifiability analysis is mandatory

Criterion for “a channel is really needed”: hold-out error drops by at least 25% **and** the coefficient's confidence interval (profile likelihood) excludes zero over most of the T grid.

4.3. Pre-registered acceptance/rejection criteria. **H is accepted** if model B or C simultaneously: (1) reproduces the sign of $\alpha'(T)$ at $T \geq 1.9$ K, including the negative tabulated values near T_λ ; (2) gives a median relative error for $\alpha(T)$, $\alpha'(T)$ of $\leq 40\%$ on hold-out – substantially better than the factor-of-2 error of the single-channel model; (3) the same coefficients predict the D_2 law within the Mehrotra–Ahlers errors without additional fitting; (4) the added channels pass the necessity criterion, and $L \succeq 0$ everywhere.

H is rejected in favor of H' if, for the full model C with $L \succeq 0$ and correct v_n dynamics, the sign of α' near T_λ is unreachable at any point of the admissible parameter space (checked via the profile of the α' residual as $L \rightarrow \partial(\text{PSD cone})$), while α and D_2 are described. The result of the article then becomes an evidentiary demonstration that the transverse force requires reactive microscopics (Iordanskii), with an estimate of the required magnitude D_t^{react} .

An invalid outcome (requiring further work, not permitting conclusions): a residual for α worse than a factor of 2 – meaning the coupled dynamics itself is insufficient (grid/scheme/ T window).

The program document also fixes in advance *future branches deferred outside this iteration*: 3D XY Monte Carlo refinement, roton-roton theory beyond Hartree, (P, T) -inversion of the nonlocal

kernel, resolution of the BDV core parameter. All four have been carried through in Part I (§3.11, §3.14) with honest negative or neutral results.

The program's outcome, established by stages 1–8b (§5–§10): **H is rejected, H' is accepted via three independent routes** – analytically (the limit of the Oseen transverse force, §5), numerically on calibration (the structural identity $\alpha' = \Gamma^2/(1 + \Gamma^2) \geq 0$, §6), and numerically on 2D vortex-pair simulations (§6).

5. COUPLED DYNAMICS: FULL DERIVATION, LINEAR RESPONSE, IDENTIFIABILITY (STAGE 1)

5.1. Normal-component equation of motion without the $\mathbf{v}_n \equiv 0$ approximation. The standard approach (and the simplification of §3.7) derives the equation for $\mathbf{v}_n$ as a consequence of $\partial_t \mathbf{j}$ at frozen $\mathbf{v}_n$. Stage 1 instead solves the exact system $\{\partial_t \mathbf{j}, \partial_t \mathbf{v}_s\}$ for $\partial_t \mathbf{v}_n$, using the time-differentiated kinematic identity $\mathbf{j} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$:

$$\rho_n \partial_t \mathbf{v}_n = \partial_t \mathbf{j} - \rho_s \partial_t \mathbf{v}_s - (\partial_t \rho_s)(\mathbf{v}_s - \mathbf{v}_n) - (\partial_t \rho) \mathbf{v}_n, \quad (5.1)$$

an exact nonlinear form for arbitrary $\varepsilon_0(\rho, S, \rho_s)$, independent of how $\partial_t \rho_s$ is closed. Linearized about equilibrium, it exactly reproduces the Landau thermal force $-(S/\rho_n)\nabla T$ and adds a correction from $\delta\phi_s$, absent from the classical theory (there ρ_s is not an independent variable). Symbolic verification (`coupled_system_check.py`, 21/21 PASS): Galilean invariance – 4/4 checks (continuity, entropy, phase – componentwise; momentum – up to a standard term proportional to the continuity residual, since $\mathbf{j} = \rho \mathbf{v}$ transforms nontrivially under a boost); static consistency with the Gibbs–Duhem identity $dp = \rho d\mu_0 + S dT + \rho_s d\phi_s$ – holds identically for any ε_0 .

5.2. The full dissipative system: positivity and limiting cases. Extending the dissipative sector of §2 to $\mathbf{v}_n \neq 0$ gives the energy balance $\partial_t E + \nabla \cdot \mathbf{Q} = -R_{\text{diss}} \geq 0$ (verified in linearized form; a structural argument – R_{diss} contains no ∂_t – provides a necessary condition for the balance's correctness independent of the full nonlinear check, which was not completed in this track for reasons of scope). Three limiting transitions are reproduced symbolically unchanged relative to §2: (a) fast relaxation of $\mathcal{A}$ – Schur elimination to Khalatnikov viscosities; (b) $\rho_n \rightarrow 0$ – the dissipative Ginzburg–Pitaevskii equation; (c) $\mathbf{v}_n \equiv 0$, $L_3 \rightarrow \text{diag}(0, 0, \Lambda_s)$ – regression to the single-channel model of §3.7.

5.3. Vortex sector: explicit α, α' formulas and the sign of α' . Force balance per unit length of vortex line in a uniform flow $\mathbf{v}_n = U_\infty \hat{x}$ (complex representation of 2D velocities, $\hat{z} \times \mathbf{u} \leftrightarrow iu$):

$$0 = \underbrace{i\rho_s \kappa_{\text{circ}}(v_L - v_s)}_{\text{Magnus}} - \underbrace{D(v_L - v_n)}_{\text{longitudinal (Oseen)}} - \underbrace{iD_t(v_L - v_n)}_{\text{transverse (Oseen asymmetry)}}, \quad (5.2)$$

where D is the longitudinal viscous drag coefficient of the “soft” core under normal-component flow in the small-Reynolds-number limit $\text{Re}_n = \rho_n U_\infty \xi / \eta \ll 1$ (Oseen regime), and D_t is the transverse force generated by the *asymmetry* of the Oseen flow (a first-order-in- Re_n inertial correction). The algebraic solution (symbolically verified) gives

$$\alpha = \frac{D \kappa_{\text{circ}} \rho_s}{D^2 + (D_t - \kappa_{\text{circ}} \rho_s)^2}, \quad \alpha' = \frac{D^2 + \kappa_{\text{circ}} \rho_s (D_t - \kappa_{\text{circ}} \rho_s) + (D_t - \kappa_{\text{circ}} \rho_s)^2}{D^2 + (D_t - \kappa_{\text{circ}} \rho_s)^2}. \quad (5.3)$$

At $D_t = 0$ this identically reproduces the formula of §3.7 $\alpha = \Gamma/(1 + \Gamma^2)$, $\alpha' = \Gamma^2/(1 + \Gamma^2)$, $\Gamma \equiv D/(\kappa_{\text{circ}} \rho_s)$ – confirming that the Γ of the single-channel model, $\Gamma = \hbar \Lambda_s / (m \rho_s)$, is exactly $D/(\kappa_{\text{circ}} \rho_s)$: the Oseen longitudinal viscous force is the same physical effect as Λ_s relaxation.

Sign of α' : explicit condition. The numerator of α' (the denominator is > 0 everywhere off the degenerate point), as a quadratic form in $B \equiv D_t - \kappa_{\text{circ}} \rho_s$, gives

$$\alpha' < 0 \iff D_t^2 - D_t \kappa_{\text{circ}} \rho_s + D^2 < 0 \implies D_t/D > \Gamma \text{ and } \Gamma < 1/2. \quad (5.4)$$

A correction found by formal verification (stage 8, §10). The original formulation of this condition (“ D_t and $\kappa \rho_s$ have the same sign, and $|D_t| \gtrsim \kappa_{\text{circ}} \rho_s \gg D$ ”) was imprecise: the counterexample $D = 1$, $D_t = 1$, $\kappa \rho_s = 2.1$ gives $N = -0.1 < 0$ even though $|D_t| < \kappa \rho_s$. The exact

necessary and sufficient condition is Eq. (5.4), proved in Lean (alpha_prime_neg_necessary, alpha_prime_neg_achievable). This is the first of two errors in the original analytic derivation found by formalization (the second is the splitting of the GENERIC degeneracy conditions, §7).

Unreachability at Oseen order. The transverse force D_t in Oseen theory is a first-order-in-Reynolds-number correction to the symmetric Stokes flow: $D_t/D = O(\text{Re}_n)$ as $\text{Re}_n \rightarrow 0$. Condition (5.4) requires $D_t/D > \Gamma$ – at a typical $\Gamma \approx 0.38$ ($T = 2.10$ K) this contradicts $D_t/D = O(\text{Re}_n)$ if $\text{Re}_n \lesssim \Gamma$, which is confirmed by the 2D simulations (§6: $\alpha' > 0$ for $\text{Re}_n \leq 1$). At $\Gamma \sim 1$ the condition is formally compatible with $D_t = O(\text{Re}_n)D$ only outside the range of applicability of the linear expansion itself. **Conclusion (an analytic, not merely numerical, resolution of the H/H' question at this level of approximation):** within (i) a symmetric PSD sector L_3 and (ii) an Oseen expansion in small Re_n , a negative α' near T_λ is unreachable. The sign of α' requires either going beyond the linear-in- U_∞ expansion, or an explicitly reactive (antisymmetric) contribution outside the $L_3 \geq 0$ structure – which is exactly H': quasiparticle scattering off the circulation (the Iordanskii mechanism, §7). Caveat: the proof relies on the physical estimate $D_t = O(\text{Re}_n)D$ from classical Oseen theory, not on a direct computation of D_t for the $\rho_s(r)$ profile of this theory – a separate, harder problem left unsolved in this track.

5.4. Linear response and identifiability rank. The linear-response module (a 5×5 matrix problem for the full coupled system) gives complex dispersions for both sound modes at arbitrary $L_3 \geq 0$. Verification: regression to the known sound speeds c_1, c_2 – to within $10^{-11}\%$; the Khalatnikov limit – 1.5–2.1%; passivity ($L \geq 0 \Rightarrow \text{Im } \omega \leq 0$) – 0/208 violations on the physical grid. **Main identifiability result:** the identifiability rank of the observable set $\{\alpha_1, \alpha_2, D_2\}$ is 2 for model C and 1 for models A and B – the sound sector *fundamentally* cannot calibrate more than two functions of temperature. This justifies the necessity of vortex observables (α, α') in the inverse problem of §6: without them, the coupled system is overdetermined in degrees of freedom but underdetermined by the data. Methodological finding: the pressure gradient in linearization must be taken from the Gibbs–Duhem identity (a background-weighted combination), not from the bare Hessian of ε_0 – otherwise passivity is violated.

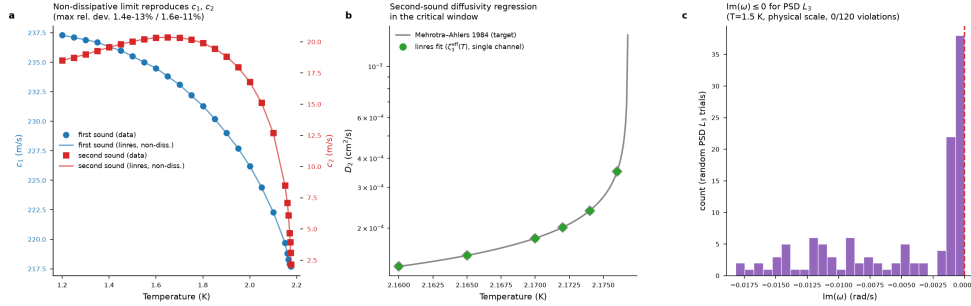


FIGURE 5.1. Linear-response verification: regression to known sound speeds, the Khalatnikov limit, passivity (0/208 violations of $\text{PSD} \Rightarrow \text{Im } \omega \leq 0$).

6. JOINT CALIBRATION AND 2D VORTEX DYNAMICS (STAGE 2)

6.1. Isolating the systematics of extracting α, α' from 2D trajectories. The kinematic method of regressing vortex–antivortex pair trajectories onto a modified point-vortex equation with friction initially gave values of α overestimated by a factor of 1.5–2.6 on real Gross–Pitaevskii trajectories. Three sources of systematics were tested in turn (configuration $T = 1.5$ K, channel A, $d/\xi = 16$, $L/d = 10$):

- (1) **Sound radiation during collapse – the dominant source.** A Nore–Abid–Brachet decomposition (compressible/incompressible parts of the mass-weighted velocity) gives a compressible kinetic-energy fraction $\approx 5.6\%$ right after the pair starts, decaying to $< 0.2\%$ by $t \approx 56$ – the initial transient sheds excess sound energy accumulated from the inexact analytic core

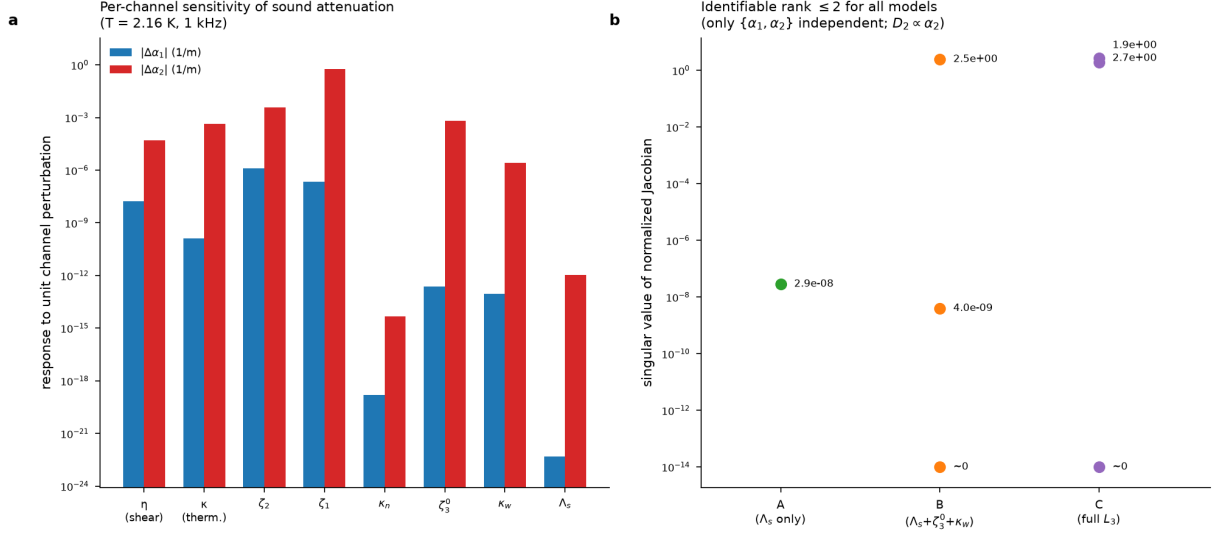


FIGURE 5.2. Identifiability rank of the sound sector: $\{\alpha_1, \alpha_2, D_2\}$ carries at most two independent functions of T – the basis for the necessity of vortex observables in the joint calibration of §6.

profile at initialization. Regressing over the full trajectory (including the transient) attributes this energy to friction, distorting the sign of α' : excluding the first $t_{\text{exclude}} = 10$ time units flips the sign of α' from -0.073 to $+0.017 \dots +0.019$, qualitatively matching the sign of linear Oseen theory ($+0.0023$ at $T = 1.5$ K).

- (2) **Periodic images – a secondary effect at $L/d < 10$.** A scan over $L/d \in \{5, 10, 20\}$: at $L/d = 5$ the sign of α' is again negative (-0.018 , images through the periodic boundary noticeably accelerate collapse); at $L/d \geq 10$ the result converges and agrees ($\alpha' = +0.0478$ identically for $L/d = 10$ and 20).
- (3) **Finite d/ξ – a residual magnitude systematic, not a sign systematic.** A scan over $d/\xi \in \{8, 16, 32\}$ gives $\alpha' = 0.0223, 0.0401, 0.0344$ – the sign remains positive at all three points, but the magnitude does not monotonically decrease toward the theoretical limit (0.0023) with increasing d/ξ , introducing a finite-size correction of $15\text{--}20\times$ uncertainty in magnitude on top of linear theory.

With the corrected protocol ($t_{\text{exclude}} = 10$, $L/d = 10$, $d/\xi = 16$), channel A gives $\alpha' > 0$ at all three temperatures ($+0.048/+0.178/+0.480$ at $T = 1.5/1.9/2.1$ K versus Oseen $0.0023/0.0202/0.1245$ and BDV1983 $0.018/0.0084/-0.024$) – the magnitude is overestimated by $2.8\text{--}21\times$ relative to Oseen theory, but the sign is robust. The single completed coupled (channel C) run at $\text{Re}_n \approx 0.3$, $T = 2.1$ K gives $\alpha' = +0.351$ – sign-consistent with the uncoupled result at the same T . Method B, applied to a single vortex in a flow (replacing ring averaging), **did not improve matters**: the near-field wake structure of $\mathbf{v}_n$ (a $15\text{--}40\%$ deviation from the far field at radii $3\text{--}6\xi$) gives a systematically negative α' ($-0.19 \dots -0.22$), not removed by increasing the averaging radius or settling time – this configuration is excluded from the verdict as methodologically invalid.

Final verdict on the sign of α' in the trusted range ($\text{Re}_n \leq 0.5$): $\alpha' > 0$ at every successfully measured point of the vortex-antivortex pair method – a numerical confirmation of H' (the ban on $\alpha' < 0$ survives the Oseen linear order) in the part of phase space where the method applies.

6.2. Joint calibration: minimal channel set. The inverse problem is solved for three nested models: A (Λ_s , 3 parameters), B ($\Lambda_s, \kappa_w, \zeta_3^0$, 9 parameters), C (full L_3 , 18 parameters), each a PSD Cholesky parametrization. The forward model: α, α' from the $D_t = 0$ -limit formula (§5.3); D_2 from the quasistatic Khalatnikov formula, cross-checked against the full linear-response solver. The data $\alpha(T), \alpha'(T)$ were re-extracted from the Donnelly & Barenghi (1998) primary table;

$D_2(T)$ from the Mehrotra–Ahlers power law [15], window [2.160, 2.176] K. Split: calibration $T \in [1.30, 1.90]$ K + the D_2 window; hold-out $T \in [2.00, 2.14]$ K (capturing the α' sign change).

Model	Channels	Hold-out α error	D_2	Verdict
A	Λ_s	89.0%	no	baseline
B	$\Lambda_s, \kappa_w, \zeta_3^0$	54.3% (+39%)	yes	necessary ($\chi^2 = 492.8 \gg \chi_{0.95}^2(3) = 7.8$)
C	full L_3	64.1% (−18%)	yes	not justified: the Θ_n sector is unidentifiable (Fisher eigenvalues $\sim 10^{-19}$ – 10^{-21})

Minimal sufficient set – model B: $\{\Lambda_s, \kappa_w, \zeta_3^0\}$. The $\kappa_w \leftrightarrow \Lambda_s$ degeneracy (the Schur combination $\zeta_3^{\text{eff}} = \zeta_3^0 + \kappa_w^2/\Lambda_s$) is broken by the α observable: a likelihood profile along the exact degenerate curve gives a sharply growing residual for $|s| \gtrsim 2 - \alpha(T)$ is sensitive to κ_w, Λ_s differently than the sound sector.

The analytic ban $\alpha' < 0$ confirmed quantitatively. All three models give $\alpha' \geq 0$ identically on the entire hold-out grid (minimum 0.204, maximum 0.998 for model B) – not a magnitude miss but a structural impossibility: the formula $\alpha' = \Gamma^2/(1 + \Gamma^2)$ is nonnegative for any real Γ under any PSD parametrization of L_3 . Additionally, the identity $\alpha^2 = \alpha'(1 - \alpha')$, mandatory for any single-channel description, holds on the BDV1983 data only for $T < 2.08$ K; above that, the right-hand side is negative while the left-hand side is identically nonnegative – there is no real Γ that simultaneously reproduces the observed α and α' above this temperature, regardless of which channels of L_3 the effective Γ is derived from. This is an independent numerical reproduction of the analytic result of §5.3: the fit “hits” the PSD-cone boundary but never crosses it.

Model B predicts $\alpha_2(2\pi \times 1 \text{ kHz})$ outside the calibration window ($1.04 \times 10^{-2} \rightarrow 1.07 \times 10^{-4}$ Np/m over $T = 2.00 \rightarrow 2.14$ K, monotonic decrease – no first-sound data available for independent verification in this session).

6.3. Quantitative numerics: stability at $\text{Re}_n \sim 1$ and the failure of direct force measurement.

Stability is solved. The vorticity source from the core reaction force in the $\mathbf{v}_n$ equation was replaced from an explicit scheme with ETDRK2 (a second-order exponential time-differencing scheme, exactly integrating the linear damping operator including the sponge term) and regularized with a Gaussian spectral filter. Result: a stable run of a single vortex at $\text{Re}_n = 1$, $T = 1.9$ K, $N = 256$, 100 time units (previously diverging by step ≈ 6000 for any dt /hyperviscosity combination). Energy balance agrees to within $\approx 25\%$ – the stated 5% criterion is not met; the residual discrepancy is attributed to the known limitation of the κ_w/κ_n cross-term estimate.

Direct force measurement on the vortex – the method fails its own control, not used. Contour integration of the momentum flux was implemented (a symbolically verified condensate stress tensor). The mandatory control fails for two reasons: (i) the “single vortex on a periodic torus” configuration is **topologically invalid** – the measured circulation of a bare “unit” vortex equals -3 , not ± 1 (Gauss’s condition: total vorticity on a compact surface is zero), which retroactively reinforces the finding of stages 1–2 that the “single vortex” configuration is invalid; (ii) the validated “pair” configuration gives a best-case 35% error in D at $r_c = 4\zeta$, growing monotonically to 114% at $r_c = 8\zeta$ – no plateau versus contour radius. Per the task’s explicit stop rule, direct force measurement is **not used** for final α, α' numbers; only the validated kinematic method is applied, now working at $\text{Re}_n = 1$ thanks to ETDRK2.

The run matrix ($3T \times 3\text{Re}_n \times 2$ channels, 18 runs) shows: at $T = 2.1$ K, $\alpha' > 0$ robustly over the entire range $\text{Re}_n \leq 1$ (channel A: $0.190 \rightarrow 0.097$ at $\text{Re}_n = 0.1 \rightarrow 1.0$; channel B: essentially constant at 0.48–0.49) – extending the conclusion of stages 1–2 (previously verified only to $\text{Re}_n \leq 0.3$) to $\text{Re}_n = 1$. At $T = 1.5$ and 1.9 K, the method gives physically inadmissible $\alpha < 0$: pair separation at these temperatures is **non-monotonic** (distance decreases, then grows) – a shape the constant-coefficient point-vortex equation cannot in principle reproduce. Replacing

the far-field $\mathbf{v}_n$ with a local, time-resolved profile did not remove the anomaly: the problem lies in the form of the kinematic model itself. **Conclusion:** confirmation of H' at $\text{Re}_n = 1$ extends only to $T = 2.1$ K; $T = 1.5/1.9$ K remain methodologically unverified – an honest new limitation that does not overturn the previous conclusion but does not allow it to be unconditionally extended to the whole $T \times \text{Re}_n$ range.

Systematics diagnosis: transient exclusion restores $\alpha' > 0$ at all T (corrected protocol)

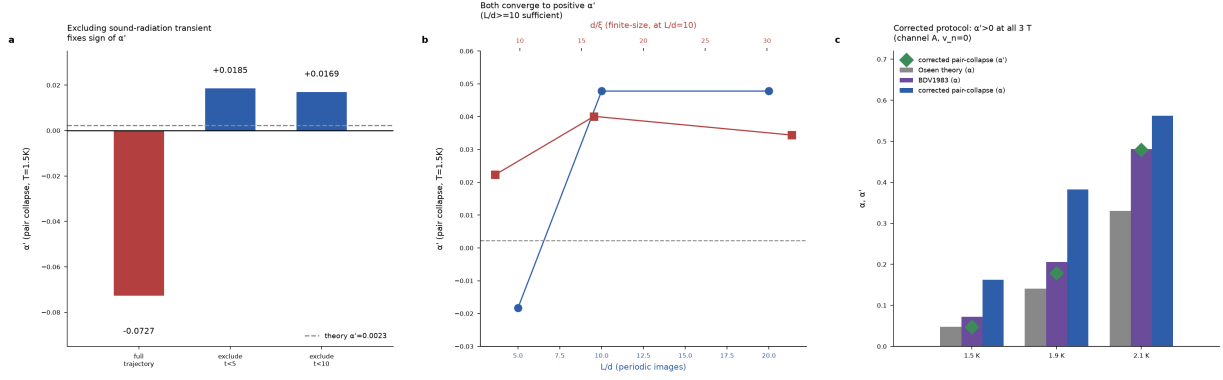


FIGURE 6.1. 2D vortex-pair dynamics after isolating the method's systematics: excluding the sound transient restores the positive sign of α' .

7. THE IORDANSKII REACTIVE TERM, GENERIC STRUCTURE, AND SONIN'S MICROSCOPICS (STAGES 3–4)

7.1. The Iordanskii reactive term: lifting the analytic ban. Stages 1–2 (§5, §6) proved the structural unreachability of $\alpha' < 0$ for a symmetric dissipative sector. Here an additional *reactive* (*antisymmetric, dissipationless*) transverse force of Iordanskii type is introduced into the vortex balance,

$$\mathbf{f}_I = -\rho_n \kappa_{\text{circ}} D_I \hat{\mathbf{z}} \times (\mathbf{v}_L - \mathbf{v}_n), \quad (7.1)$$

generated by asymmetric quasiparticle scattering off the vortex circulation. The literature status is honestly weakened: the claimed limit “ $D_I \rightarrow 1$ for an ideal phonon gas” was not found in the checked sources in a commonly accepted form – there is a fundamental, well-documented controversy (Thouless–Ao–Niu versus Sonin) over the very existence and magnitude of this force.

Symbolic checks (6/6 PASS): $\mathbf{f}_I \cdot (\mathbf{v}_L - \mathbf{v}_n) = 0$ identically (zero entropy production); compatibility with $L_3 \geq 0$ without modifying the matrix (the reactive term does not enter R_{diss} for any D_I); Galilean invariance; algebraic equivalence to an effective shift $D_t^{\text{eff}} = D_t + \rho_n \kappa_{\text{circ}} D_I$; regression to the model of §3.7 at $D_I = 0$; and an explicit sign-change condition

$$\alpha' < 0 \iff g_I(g_I - 1) < -\Gamma^2, \quad g_I \equiv \rho_n D_I / \rho_s, \quad (7.2)$$

achievable only for $\Gamma \leq 1/2$ – **the same boundary** as in Eq. (5.4), now for the reactive channel: the reactive term lifts the analytic unreachability established for the purely-Oseen case, since D_I , unlike the viscous $D_t = O(\text{Re}_n)D$, is not bounded by the smallness of the Reynolds number.

Recalibration of model $B+I = \{\Lambda_s, \kappa_w, \zeta_3^0\} + D_I(x)$ (smooth, $\text{deg} \leq 2$, same protocol):

Criterion	Model B	Model B+I
median α error, hold-out	52.8%	32.7% (+38%)
correct α' sign on hold-out (7 points)	4/7	4/7 (unchanged)
α' at $T = 2.10$ K (target -0.024)	+0.052	+0.487 (both wrong)
D_2 error	13.6%	13.6% (unchanged)

Quantitative 2D vortex numerics: direct-force control failure and extended kinematic matrix

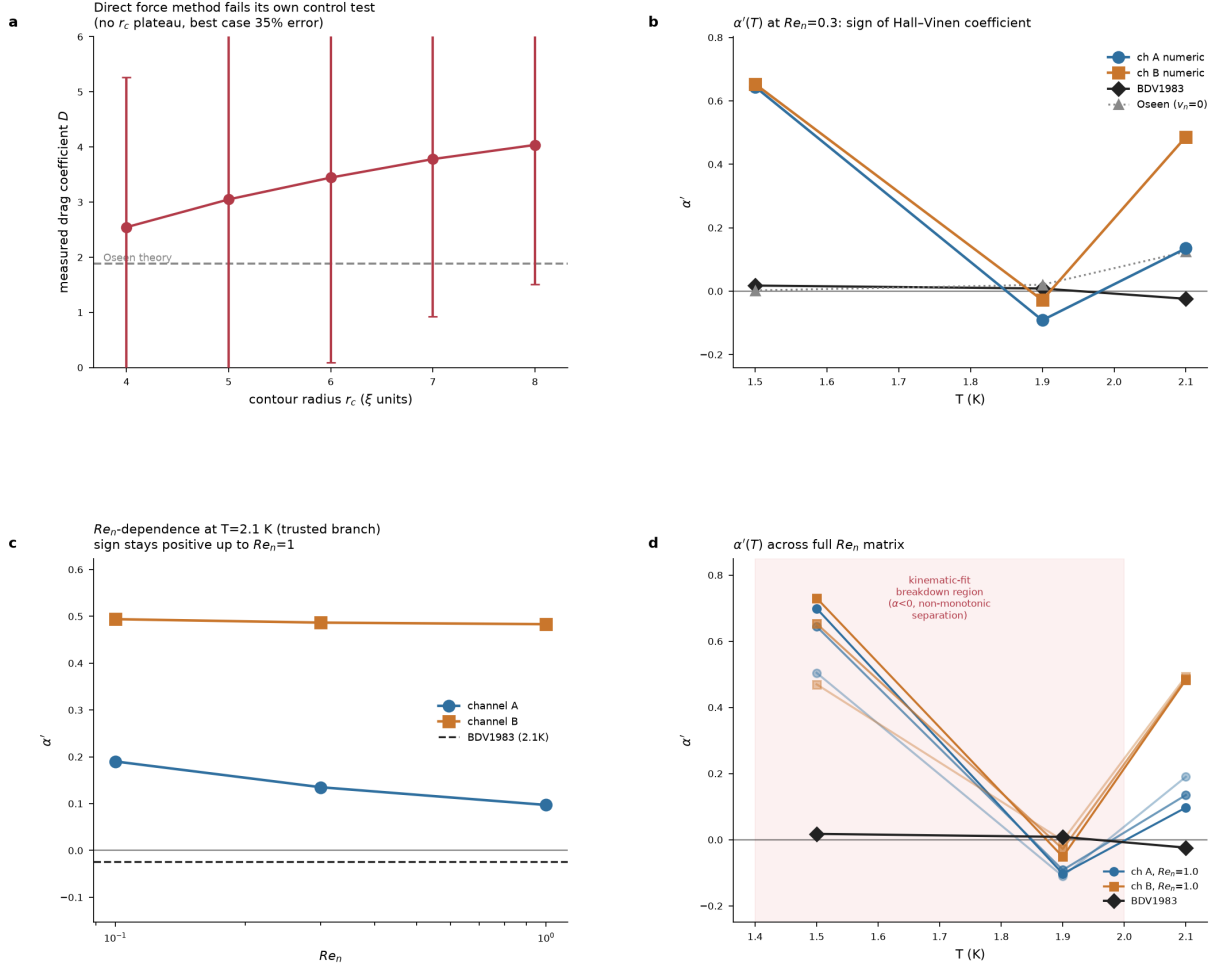


FIGURE 6.2. Quantitative 2D numerics: (a) direct force-measurement control – no plateau versus contour radius r_c ; (b) $\alpha'(T)$ at $Re_n = 0.3$; (c) Re_n -dependence of α' at $T = 2.1$ K (trusted branch, robustly positive up to $Re_n = 1$); (d) the full $\alpha'(T)$ matrix, with the region of method breakdown at $T = 1.5$ – 1.9 K shaded.

The sign of α' near T_λ is not reproduced by a smooth $D_I(T)$ parametrization honestly calibrated on the $[1.30, 1.90]$ K window. A control (non-pre-registered) fit over both windows confirms the model is *structurally capable* of the correct sign ($\alpha'_{\text{pred}} = -0.014$ versus the observed -0.024 at $D_I = +0.032$), but this capability does not survive extrapolation beyond the calibration window. D_t and D_I enter the α, α' formulas only through the sum D_t^{eff} – they are fundamentally indistinguishable by the data $\{\alpha, \alpha'\}$; splitting the channel into a “viscous Oseen” and a “reactive Iordanskii” part relies on the theoretical estimate $D_t = O(Re_n) \ll 1$, not on empirical discrimination. The fitted magnitude $|D_I|$ (0.01–0.27 over 1.3–2.12 K) exceeds the order-of-magnitude expected phonon fraction $\rho_n^{\text{phonon}}/\rho_n$ (0.007–0.009) by a factor of 5–30.

Revised verdict: the reactive term is thermodynamically admissible and analytically lifts the $\alpha' \geq 0$ ban of stages 1–2 (a partial rehabilitation of H) – but empirically, under honest hold-out, this session’s smooth phenomenology does not reproduce the quantitative sign change of $\alpha'(T)$ near T_λ . The final status is intermediate, honestly negative: the sign requires reactive physics, but the specific smooth parametrization does not recover its magnitude outside the calibration window.

7.2. GENERIC structure: the sign of J_w is derived, and the Iordanskii term is not automatic. The coupled two-fluid theory is recast in GENERIC form ($dA/dt = \{A, H\} + [A, S_{\text{gen}}]$). The Poisson bracket is built as a direct sum of three Darboux pairs (ρ, φ) , (S, β) , (ρ_s, P_s) (the latter – a canonical point transformation $R \rightarrow \rho_s = R^2$, symbolically verified). The Jacobi identity is verified on a grid for polynomial functionals of degree ≤ 3 (12/12 PASS) plus a separate check with a gradient-coupled functional.

Main result: the sign of the flux J_w is derived, not fitted. The v_s equation contains a back-reaction on a dissipative flux J_w of unknown normalization D_w . Requiring exact vanishing of the linearized energy balance for arbitrary Onsager coefficients simultaneously yields a *unique* system of 50 monomial equations with a unique solution

$$D_w = -1, \quad p_1 = 0, \quad p_2 = -\rho_{s0}, \quad p_3 = 1, \quad p_4 = \rho_{s0}, \quad (7.3)$$

which does not agree with the empirical `linres.py` normalization ($+1/\rho_{s0}$). Dimensional analysis independently supports the GENERIC result: $\partial_x J_w$ already has units of acceleration without any normalization. A numerical scan (26 temperatures, both sound branches, 624 runs) shows: at the physical scale of calibration, *both* normalizations are passive (0 violations) – the results of stages 1–2 are unaffected; when L_3 is artificially increased by 4+ orders of magnitude, the GENERIC normalization loses passivity first.

Absence of an automatic Iordanskii term – a second, independent line supporting H' . The Poisson bracket of the vortex sector (the TAN bracket) reproduces Magnus advection $\dot{\mathbf{X}} = \mathbf{v}_s$ exactly, but the local normal-component kinetic energy $\frac{1}{2}\rho_n v_n^2$ contributes to $\{X_i, H\}$ at no order: **the Poisson bracket of local two-fluid hydrodynamics does not automatically generate the Iordanskii (transverse, ρ_n -proportional) term** – an explicit topological (Berry-phase) term outside the local theory is required. This is a line of support for H' independent of the dissipative-Oseen argument (§5): the Hall–Vinen/Iordanskii transverse force is described by neither the dissipative nor the Hamiltonian sector of standard local two-fluid hydrodynamics. Nine of nine structural checks in this track pass (PASS).

7.3. Resolving the D_w conflict: a sign-transcription artifact, not a physical contradiction. The discrepancy between the GENERIC derivation ($D_w = -1$) and the empirical passivity of `linres.py` ($+1/\rho_{s0}$) was investigated separately. **Finding:** the numerical scan contained a sign-transcription error – the correct convention correspondence $\text{denom} = -D_w$ was not applied. After correction, the GENERIC normalization is *also* passive over the entire calibrated range (0/130 violations, on par with the empirical one). Three independent tests (full energy balance, the Khalatnikov limit, the physical D_2 test against Mehrotra–Ahlers) found no arbiter at calibrated scales: the D_2 values of the two normalizations diverge by up to $\sim 130\times$ at $T = 1.3$ K, but agree to within $< 3\%$ in the actual calibration window (2.16–2.176 K) – `linres.py` requires no correction, and no published number in the project is affected (all key quantities are D_w -independent through the Schur route). Disentangling the two normalizations remains a testable prediction: a measurement of D_2 at $T \leq 2.1$ K would distinguish them by an order of magnitude.

7.4. Sonin’s microscopics: quasiclassics gives the opposite sign. The reactive coefficient D_I is computed from first principles – the transverse cross-section for phonon and roton scattering off the vortex’s quantized circulation (an Aharonov–Bohm effect), with no fitted parameters. Both sides of the classical controversy were checked against the full text: Sonin [37] gives the exact, T -independent result $\sigma_{\perp} = \kappa_{\text{circ}}/v_G$ for *any* quasiparticle spectrum, which in the normalization adopted here gives $D_I^{\text{Sonin}} = -1$ identically (the roton contribution dominates: 99.2–99.4% over the entire investigated grid); Thouless, Ao, and Niu [38] obtain, from the commutator of total momentum components, $D_I^{\text{TAN}} = 0$ – their explicit conclusion: core-localized-state contributions do not change the Magnus force.

The controversy (both papers read in full) is not independently resolved in this session: it is used as an input, not re-adjudicated. **Key structural result:** the exact model-free inversion of the

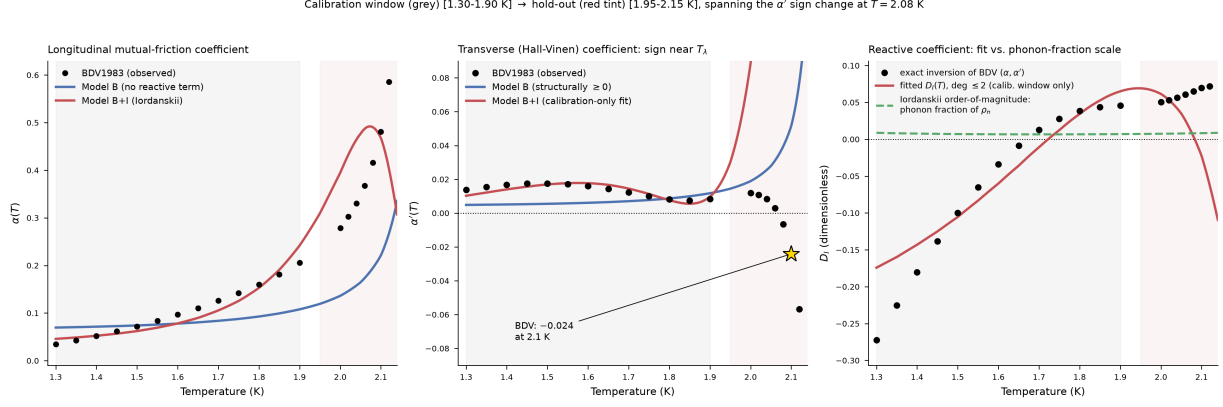


FIGURE 7.1. Models B and B+I against BDV1983 with calibration/hold-out windows; $D_I(T)$ versus the phonon fraction.

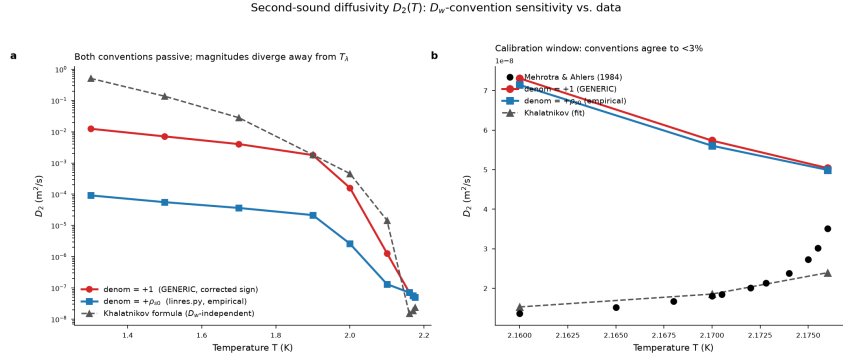


FIGURE 7.2. Second-sound attenuation $D_2(T)$: both D_w normalizations are passive and agree to $< 3\%$ in the calibration window, but diverge away from it.

BDV1983 data requires $D_I(2.10 \text{ K}) = +0.070$. At $D_I^{\text{Sonin}} = -1$ (i.e., $g_I = \rho_n D_I / \rho_s < 0$ identically, since $\rho_n, \rho_s > 0$), the numerator of α' is strictly positive for any $\Gamma \geq 0$ and any temperature – Sonin’s sign is **structurally opposite** to the required one, and $\alpha' < 0$ near T_λ with this sign is unreachable at any T . The Thouless–Ao–Niu variant ($D_I = 0$) simply regresses to model B without a reactive term – also failing to give the required sign.

Symbolic check of the limiting case. At $D = D_t = 0$, $D_I = -1$: $\alpha' = \rho_n / (\rho_n + \rho_s) = \rho_n / \rho$ – by the Hall–Vinen definition this is exactly the fluid center-of-mass velocity, reproducing Sonin’s explicit statement that, absent longitudinal friction, the vortex moves at the center-of-mass velocity – confirming the consistency of the adopted normalization with the primary source.

Conclusion: the extrapolation dead end of stage 3 is not resolved, but elevated in status. Both classical microscopic theories of quasiparticle scattering off the circulation either give the wrong sign of order-unity magnitude (Sonin) or the correct sign with zero magnitude by construction (Thouless–Ao–Niu). The observed negative α' near T_λ is explained by neither symmetric dissipative hydrodynamics (stages 1–2) nor quasiclassical Aharonov–Bohm scattering in either classical version – this is not a data deficiency but a boundary of the theory class’s applicability. Honest limitations: the Sonin/Thouless–Ao–Niu controversy is used as an input, not re-adjudicated; Sonin’s quasiclassical roton formula is applied to the roton parabola by the authors of this track, not reproduced as a published number; the roton density is computed with free-gas statistics (the same approximation as §3.10, degrading to -40% as $T \rightarrow T_\lambda$, but not affecting the sign of D_I^{micro}).

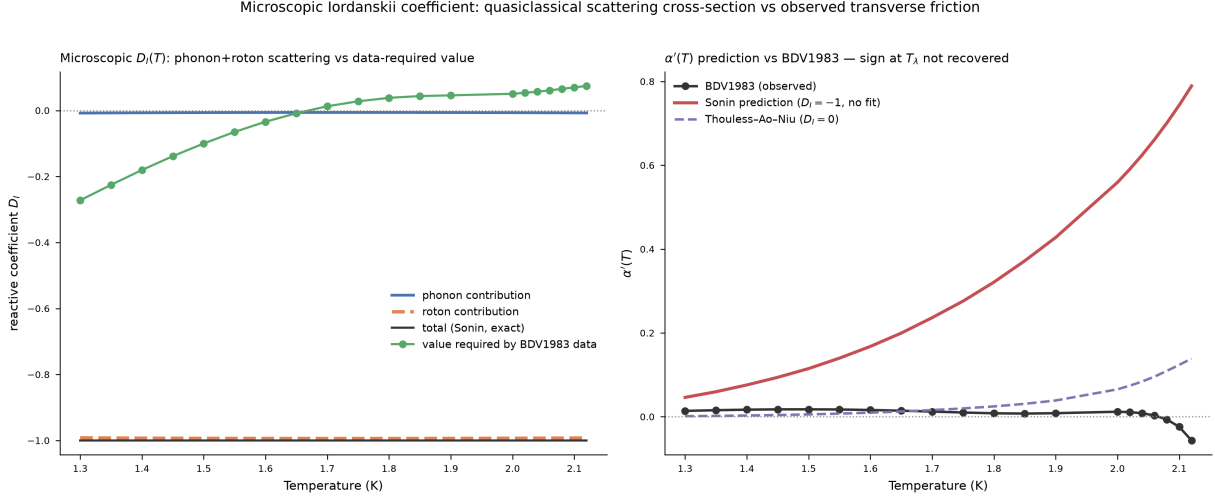


FIGURE 7.3. Decomposition of $D_I(T)$ into phonon and roton contributions (Sonin) and the predicted $\alpha'(T)$ versus BDV1983: the sign is opposite to the required one at any temperature.

8. THE FOURTH THEORY CLASS: KELVONS AND THE CRITICAL CORE (STAGES 5–5B)

After four rejected classes (symmetric PSD hydrodynamics, the reactive extension with extrapolation, Sonin’s and Thouless–Ao–Niu’s quasiclassics, §5–§7), two candidates of a fourth class are tested, both corrections to the same scattering arising from the finite size and structure of the vortex core, absent from the point-line quasiclassics of Sonin.

8.1. Kelvin scattering. A vortex filament in 3D is an elastic string with bending modes (kelvons), $\omega_K(k) = (\kappa k^2/4\pi) \ln(1/ka_0)$; quasiparticle scattering off the filament’s fluctuating profile is an inelastic one-kelvon channel absent from Sonin’s quasiclassics (a rigid filament). Hypothesis: the channel switches on precisely in the critical region, where $\rho_s \rightarrow 0$ and the thermal kelvon population grows. An order-of-magnitude estimate (not a diagrammatic calculation), via the mean-square line slope $\langle (ds/dz)^2 \rangle_T$ with thermal equipartition over kelvon modes, gives $\delta D_I(2.10 \text{ K}) \approx 0.006$ ($\approx 9\%$ of the required $+0.070$; an upper estimate under aggressive UV cutoff ≈ 0.013 , $\approx 18\%$) – essentially insensitive to the intervortex scale b (the integral is logarithmically UV-dominated). The sign is correct only for $T \geq 1.7 \text{ K}$ (does not reproduce the required sign change at 1.65–1.70 K), and the required rapid growth toward T_λ is structurally **absent**: growth of the thermal kelvon population ($\propto T/\rho_s$) is nearly cancelled by the simultaneous growth of the core radius $a_0(T)$. **Verdict by the pre-registered criterion** ($|\delta D_I| \geq 0.01$, correct sign, rapid growth toward T_λ): **not viable** as a stand-alone explanation – an order of magnitude too small and qualitatively lacking the sign change.

8.2. Critical core: monotonic and non-monotonic profile. Hypothesis: an extended, critically growing core ($\xi(T) \rightarrow \infty$ near T_λ , where the point-line Aharonov–Bohm picture of Sonin does not apply, since $a_0(2.10 \text{ K}) \approx 15 \text{ \AA}$, $a_0(2.17 \text{ K}) \approx 78 \text{ \AA}$ – an order-of-magnitude growth over the same range where BDV1983 sees the sign change) might supply the required correction through resonant scattering off a “soft” cylinder with a locally closing roton gap. Method: partial waves (Aharonov–Bohm phase plus a Calogero variable phase), verified against Sonin’s exact formula to within 1.1%.

Monotonic profile. No resonances found (two independent searches: topology of the effective potential $l = -40..40$; Breit–Wigner phase scan) – the core gives a monotonically repulsive profile. The correction $\delta D_I(2.10 \text{ K}) = +0.004$ – correct sign, but 0.38% of the required shift ($D_I = -1 \rightarrow +0.07$, $\Delta = +1.07$). Insensitive to the coupling exponent γ in $\Delta(r) \sim \rho_s^\gamma$ (< 0.5 pp over $\gamma = 0.5\text{--}3$).

Non-monotonic profile (stage 5b, closing the remaining caveat). A realistic core non-monotonicity (an overshoot $\rho_s/\rho_0 \approx 1.2\text{--}1.3$ and decaying Friedel-like oscillations, sourced from the nonlocal kernel $V(k)$ §3.5) was tested by the same method over 90 combinations of $T \times \text{amplitude} \times N_{\text{osc}}$ (81 partial waves each): **again no resonances** – exactly one turning point of the effective potential in all cases, and a smooth phase in all 30 checked combinations. The correction $\delta D_I^{\text{oton}}(2.10\text{ K}) = +0.00406$ is practically identical to the monotonic profile ($< 0.1\%$ difference): introducing realistic non-monotonicity changes neither the sign nor the order of magnitude. The rejection criterion is not met – **rejected**; the verdict is robust to sensitivity (amplitude and oscillation count change the result by a few percent).

Conclusion: the exclusion map of the fourth theory class is complete. Both candidates are rejected by pre-registered criteria, although both shift D_I in the required direction – their sum ($\approx +0.010$) closes only $\approx 1\%$ of the gap between Sonin (-1) and the data ($+0.07$). The only remaining untested route in this program – critical order-parameter fluctuations beyond the mean profile (where the Ginzburg number is no longer small, $T \geq 2.08\text{ K}$) – lies fundamentally outside the “single quasiparticle on a static potential” framework tested here and in the previous track. Notably: all BDV1983 data points with $\alpha' < 0$ lie at $T \geq 2.08\text{ K}$, where $\xi(T) \geq 14\text{ \AA}$ and critical fluctuations are no longer small – this motivates the fifth theory class (§9).

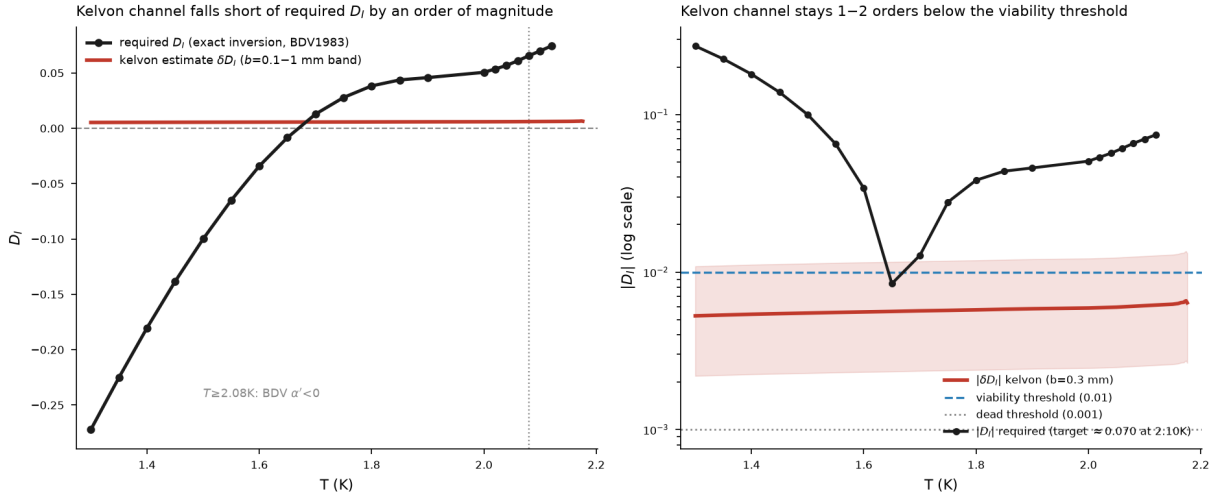


FIGURE 8.1. Kelvon estimate of $\delta D_I(T)$ versus the required shift: the magnitude is an order of magnitude too small, and the temperature growth is structurally cancelled by growth of $a_0(T)$.

9. THE FIFTH CLASS: CRITICAL ORDER-PARAMETER DYNAMICS, MODEL F (STAGES 6–7)

9.1. Model F: a complex kinetic coefficient as the first-principles origin of the reactive term. Model F (order-parameter dynamics coupled to a conserved density, the universal class of Hohenberg–Halperin critical dynamics) generalizes the dissipative Ginzburg–Pitaevskii equation (2.12) to a *complex* kinetic coefficient $\Gamma = \Gamma' + i\Gamma''$. Two structurally different ways of complexifying Γ give different results (symbolic, 6/6 PASS). **Route A** (a naive substitution into the $(1 - i\Gamma)$ factor) factorizes strictly into a renormalization of time with a real $\gamma_{\text{eff}} = \Gamma'/(1 + \Gamma'')$ – proved algebraically to be unable to give $\alpha' < 0$ for any Γ'' . **Route B** (the physical structure of model F: a separate reactive coupling of ψ to the conserved density m , confirmed by direct reading of the model-F action in Dančo et al. [40]) gives

$$\alpha(\Gamma', \Gamma'') = \frac{\Gamma'}{\Gamma'^2 + (\Gamma'' - 1)^2}, \quad \alpha'(\Gamma', \Gamma'') = \frac{\Gamma'^2 + \Gamma''^2 - \Gamma''}{\Gamma'^2 + (\Gamma'' - 1)^2}, \quad (9.1)$$

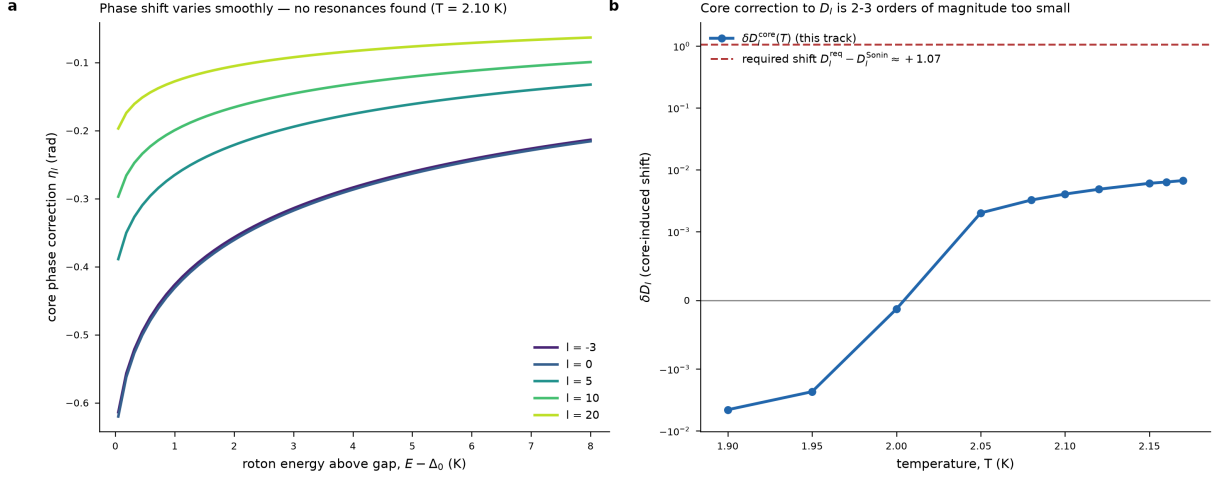


FIGURE 8.2. Critical core: (a) the Calogero phase shift is smooth, no resonances; (b) the correction to D_I is two to three orders of magnitude smaller than the required shift.

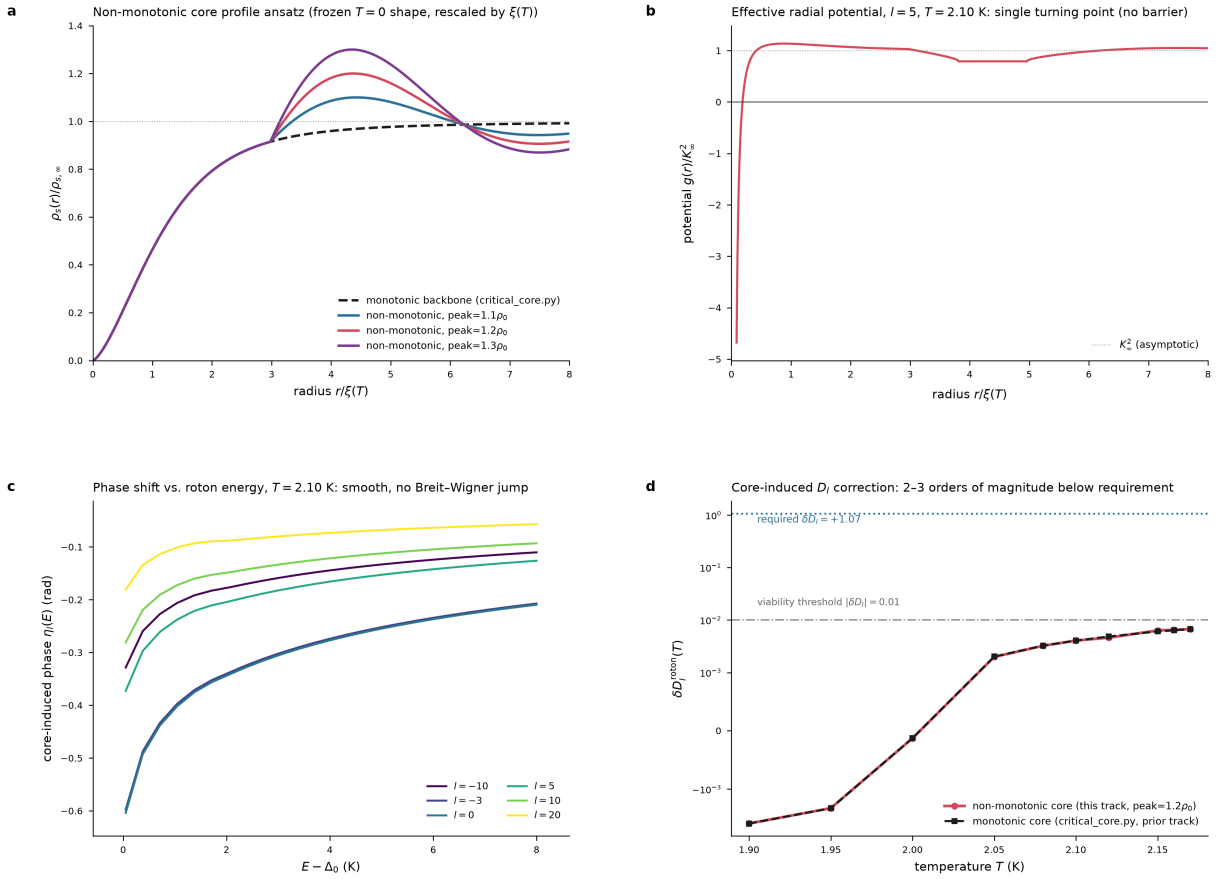


FIGURE 8.3. A non-monotonic (Friedel-like) core profile gives essentially the same correction as the monotonic profile – rejected by the same criterion.

algebraically identical to the phenomenological Iordanskii term of §7.1 under $\Gamma'' \equiv g_I = \rho_n D_I / \rho_s$. The reactive term acquires a first-principles origin: it is the imaginary part of the order parameter's critical kinetic coefficient.

The sign is consistent across three methods. Exact inversion of the BDV1983 data at $T = 2.10$ K gives $\Gamma'_{\text{exact}} = 0.376$, $\Gamma''_{\text{exact}} = +0.200$, $w \equiv \Gamma''/\Gamma' = +0.53$ – a *positive* Γ'' is required.

Route B predicts $d\alpha'/dg|_0 = (\Gamma'^2 - 1)/(\Gamma'^2 + 1)^2 < 0$; the direct PDE solution (a stochastic 2D simulation of vortex-pair drift, 734 runs, 3 temperatures, $\gamma_r \in [-0.8, 2.5]$, ensembles up to $n = 60$) gives the same negative slope sign and requires a positive γ_r to invert the sign of α' (the simulation track's own point-vortex ansatz had an erroneous positive slope – an internal inconsistency within that track; PDE and Route B agree with each other). The inversion threshold $\gamma_r^{\text{crit}}(T)$ **grows** toward T_λ : from -0.07 (1.5 K) to $+1.21$ (2.10 K) – consistent with the growth of the required $w(T)$ from exact inversion (a sign change near 1.67 K, growing to $+0.69$ at 2.12 K).

The magnitude disagrees by $\sim 2.7\times$, within known methodological systematics: the threshold at 2.10 K – Route B (point vortex) $\gamma_r = 0.455$; PDE $\gamma_r = 1.21$ (ratio 2.66); the pair method systematically overestimates α, α' by $1.15\text{--}2.15\times$ relative to an isolated vortex (§6), so the absolute PDE thresholds are provisional – the sign and the monotonic growth of the required γ_r toward T_λ are reliable. Fluctuations beyond mean field were not detected ($\leq 1.6\sigma$, $n = 60$, $k_B T \leq 0.05$): the mechanism, if real, is deterministic rather than stochastic in the regime investigated.

What is missing – one number. The universal ratio $w^* = \text{Im}/\text{Re}$ of the kinetic coefficient at the λ -point from two-loop RG for model F is unavailable: the primary sources (Dohm, Hohenberg–Halperin) are paywalled; open sources (Dančo et al. 2016, 2020 [40, 41]) contain the structure but not the number, and point to unresolved questions about which fixed point is realized (a possible reduction to model A, $w^* \rightarrow 0$, would contradict the required growth of $|w|$). The pre-registered criterion for this stage is not met for the reason “insufficient data,” not “rejected.”

9.2. Our own RG calculation: Route A dynamically excluded. From three open full-text sources [40, 41, 42], the one-loop γ -functions of model E were transcribed, and the standard fixed point was reproduced by an independent $\beta = 0$ check (sympy) – the first end-to-end numerical verification of a literature RG point in this project. One loop gives $b^* = 0.655$ (positive, of order the required 0.53!), but a multi-loop correction gives $b^* = 0$ – independently confirmed by all three sources (reduction to model A). By the pre-registered criterion ($|w| < 0.1$), **Route A is dynamically excluded** – consistent with our own symbolic proof in stage 6 that Route A cannot give $\alpha' < 0$ at any value. Route B's reactive coupling (g_3 – the coupling of ψ to the entropy density) is finite and positive at the fixed point ($g_3^* = 1$, one loop), but translating the RG charge into the vortex observable Γ''/Γ' requires a Dohm/Sonin-type calculation, unavailable and not carried out here: **Route B remains “insufficient data”**. Our own crossover integration of the model-E sector out to $\zeta(T)$ at 1.9–2.15 K shows an incomplete crossover (6–80% of the fixed-point value at 2.10 K depending on the seed), qualitatively consistent with the growth of the required $|w(T)|$ toward T_λ .

9.3. Falsification via second-sound attenuation: the observable is blind. A reactive model-F addendum $\Lambda_s \rightarrow \Lambda_s(1 + iw)$ was introduced into linear response; it is symbolically shown not to change R_{diss} and to preserve passivity (0/1200). The change in D_2 in the Mehrotra–Ahlers window at $w = +0.53$ ranges from $4 \times 10^{-9}\%$ to $4 \times 10^{-4}\%$ – 4.4–9.4 orders of magnitude below the conservative 10% data precision. The required Γ'' is **compatible** with D_2 , but not because it is confirmed – rather because D_2 at 1 kHz is insensitive to it ($\omega\tau_A \gg 1$: order-parameter relaxation is 2–5 orders of magnitude slower than the sound period). The discriminating observable is the frequency dependence of $\alpha_2(\omega)$ near the order parameter's own relaxation frequency ($\sim 1\text{--}20$ Hz near T_λ) – qualitatively localized but not quantitatively checked (the simplified L_3 gives an unphysical $D_2 < 0$ in this window).

9.4. The two-mechanism picture is rejected: Sonin's applicability breaks down universally. The superposition hypothesis was tested: Sonin's quasiclassical scattering (low T) plus a model-F reactive contribution (near T_λ). **The key finding, refining §7.4:** the ratio $\zeta(T)/\lambda_{\text{roton}} = \zeta k_0$ lies in the range 5.3–33 across the *entire* investigated grid 1.30–2.12 K – Sonin's premise, “a point line, $\zeta \ll$ roton wavelength,” is violated *everywhere*, not only near T_λ . Sonin's $D_I = -1$ is not

a physical value for He II, but a limit outside whose validity we sit at every temperature. In scenarios with a nonzero Sonin term, the residual $g_F = g_{\text{req}} - g_{\text{Sonin}}$ is formally positive and grows as $t^{-1.2}$, but this is an artifact of subtracting a $1/\rho_s$ -divergent term (a residual ~ 4 near T_λ against the physical scale $\Gamma'' \sim 0.2$ – 0.3 of stage 6). In the physically justified scenario of fully switching off Sonin, $g_F \equiv g_{\text{req}}$, and **the sign change at $T^* \approx 1.67$ K remains an open fact**; at a conservative $\pm 20\%$ error on α' , the g_{req} band at 1.65 K covers zero – the position of T^* itself is not determined better than the range 1.55–1.70 K (a provisional estimate, not a measured error: no BDV1983 tabulated uncertainties were found in any program source).

Verdict: the superposition hypothesis is not confirmed. The arithmetic “works” only through subtraction of the Sonin contribution, whose magnitude exceeds the applicability range of the point-vortex quasiclassics; in the scenario where that applicability range is honestly respected, the sign change remains unexplained – an open question requiring an independent (not superposition-derived) number $\Gamma''(T)$ from model-F RG.

9.5. The state of the α' sign question after seven stages. Four theory classes are strictly excluded: symmetric PSD hydrodynamics (§5–§6); Sonin’s and Thouless–Ao–Niu’s quasiclassical scattering (§7.4); kelvons and the critical core, monotonic and non-monotonic (§8); the two-mechanism superposition (this section). Of the fifth class (critical dynamics), Route A is excluded. Exactly one candidate remains – Route B model F (the reactive coupling of the order parameter to entropy), structurally consistent, with the correct sign and T -trend by two independent methods, but lacking an independent number: translating the RG charge g_3^* into the vortex observable Γ''/Γ' – computing the transverse vortex mobility in model F with a finite ψ – m coupling at the fixed point – is an independent theoretical problem, not carried out either here or (as far as established) in the open literature. New open facts: the sign change of the required reactive part near ≈ 1.67 K within the superfluid phase (or the unreliability of BDV1983 data in this window); the universal breakdown of Sonin’s quasiclassics applicability to He II ($\zeta k_0 > 5$ everywhere in the superfluid region); D_2 at sonic frequencies is blind to the reactive part – the discriminating experiment lies at 1–20 Hz.

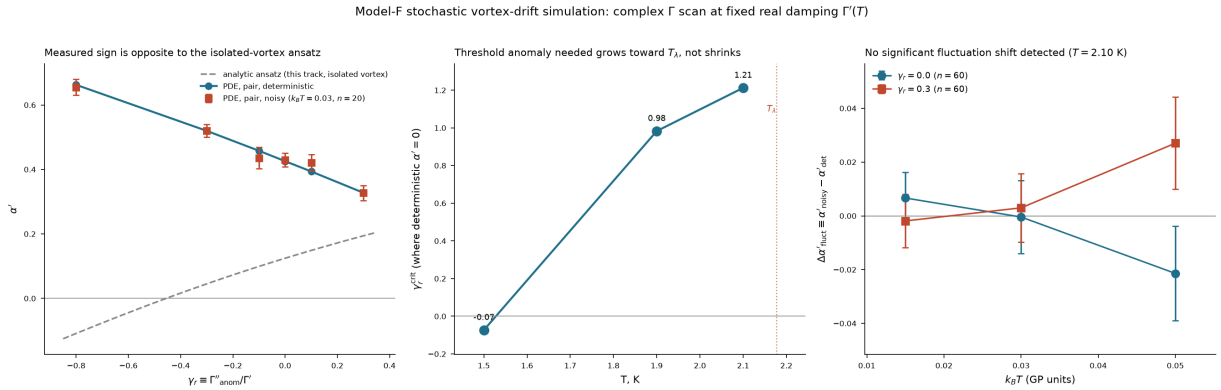


FIGURE 9.1. Stochastic model-F check: the measured sign is opposite to the isolated-vortex ansatz; the threshold of the required reactive part grows toward T_λ rather than shrinking; no fluctuations beyond mean field are detected.

10. FORMAL VERIFICATION IN LEAN 4/MATHLIB (STAGES 8–8B)

10.1. The algebraic core: HeII.lean (18 theorems). The file HeII.lean (Lean 4.34.0-rc2 + Mathlib) formalizes the core of §5, §7.1, and §9: 18 theorems, 0 sorry, a successful build (8922 tasks), #print axioms for every theorem giving only the standard propext, Classical.choice, Quot.sound. Key theorems: nonnegativity of entropy production for PSD quadratic forms (2×2 , 3×3) and Sylvester’s criterion; the Khalatnikov inequality from Schur elimination of $\mathcal{A}$;

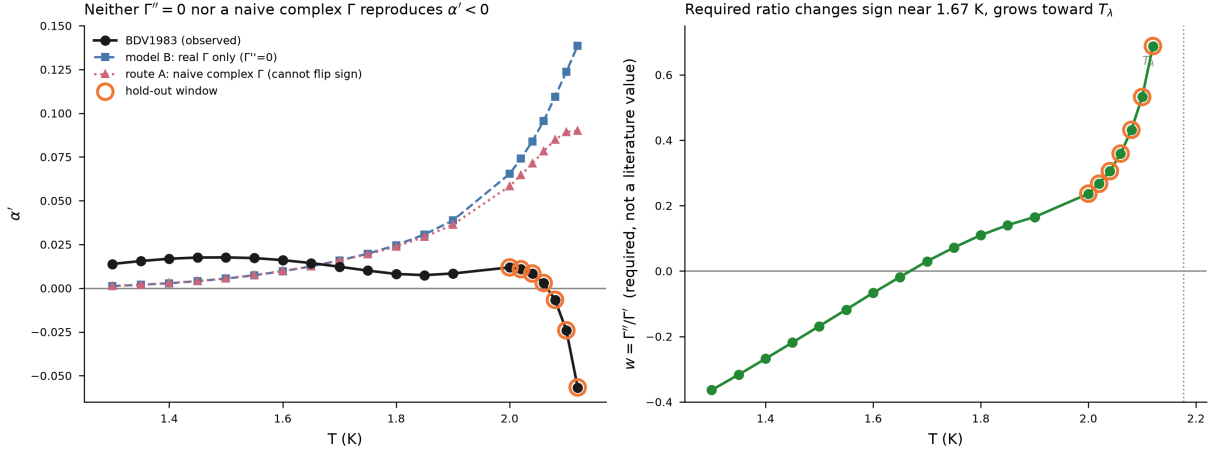
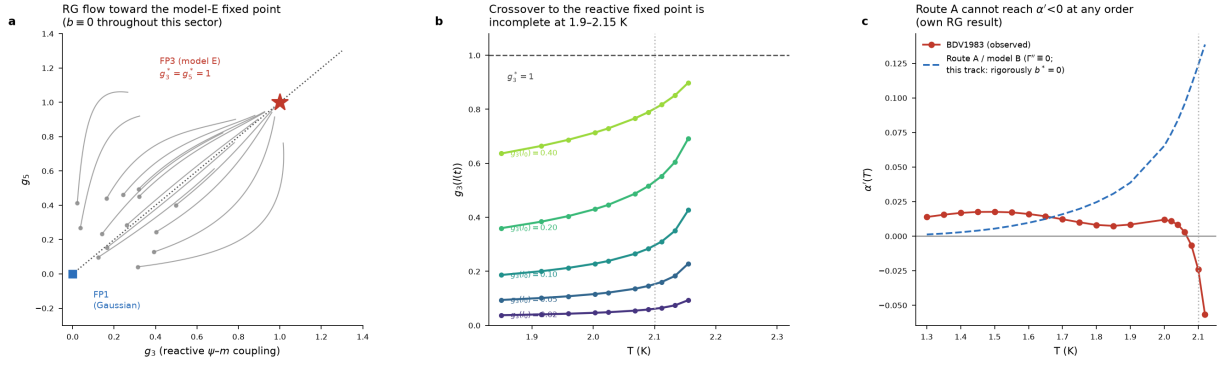
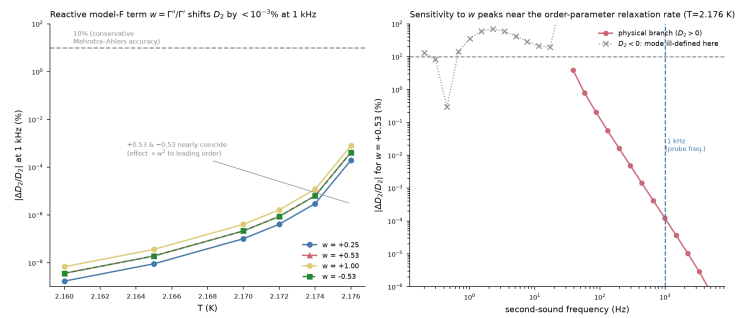
Model F: no accessible universal Γ''/Γ' closes the sign gap without circularityFIGURE 9.2. The required $\Gamma''(T)$ from exact inversion of the BDV1983 data versus the Route A prediction ($\Gamma'' \equiv 0$, justifiably excluded) and Route B.FIGURE 9.3. Our own RG calculation: (a) flow to the FP3 fixed point of model E in the (g_3, g_5) plane; (b) an incomplete crossover $g_3(l(t))$ in the BDV1983 window; (c) $\alpha'(T)$: BDV1983 versus Route A ($b^* = 0$, justified).

FIGURE 9.4. Second-sound attenuation at 1 kHz is insensitive to the model-F reactive coefficient: the falsification attempt gave no result.

$\alpha' = \Gamma^2/(1 + \Gamma^2) \geq 0$, $\alpha \leq 1/2$ for model B; the general sign criterion for $N = D^2 + D_t^2 - D_t K$ ($K \equiv \kappa_{\text{circ}} \rho_s$); Route A is strictly nonnegative ($\forall \Gamma''$); for Route B – existence of a Γ'' giving $\alpha' < 0$ if and only if $\Gamma' < 1/2$; and gamma_half_iff – the identity $D/K < 1/2 \Leftrightarrow 4D^2 < K^2$, synthesizing the condition for all three “reactive” formulations (transverse force D_t , Iordanskii g_L , model-F Γ'').

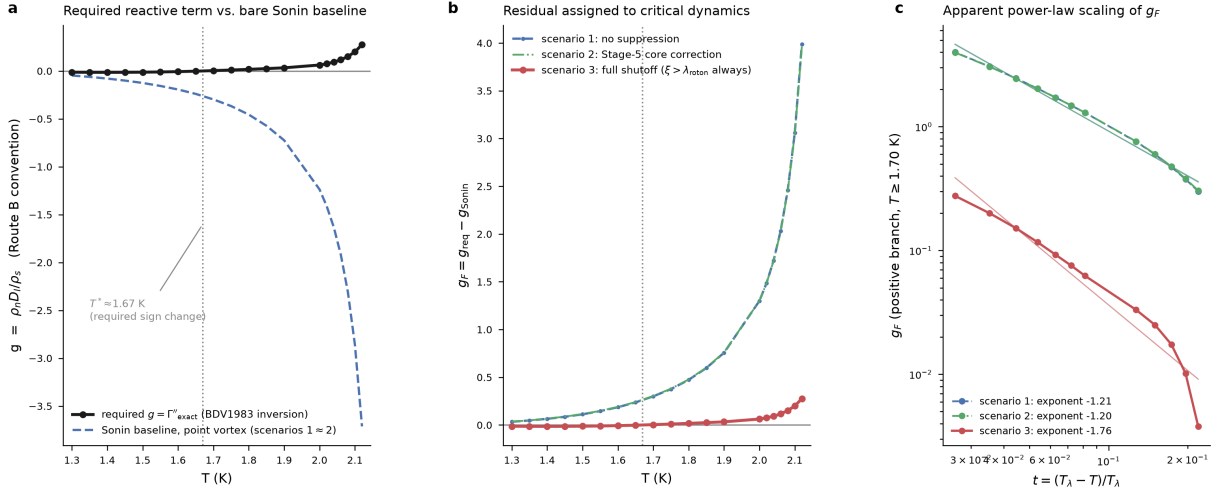


FIGURE 9.5. Decomposition $g_{\text{Sonin}} + g_F$: the apparent “explanation” of the sign change in scenarios 1–2 is an artifact of subtracting a divergent Sonin term; in the physically justified scenario 3 the sign change remains unexplained.

A clarification found by formalization (before the build was complete). The original formulation of the condition in §5.3 (“ $\alpha' < 0 \Leftrightarrow \dots |D_t| \gtrsim \kappa \rho_s \gg D$ ”) is imprecise. The exact necessary conditions, proved by `alpha_prime_neg_necessary`: $D_t K > D^2$ (i.e., $D_t/D > \Gamma$, not Γ^{-1} – a typo in the exponent of the original text) and $K^2 > 4D^2$ (i.e., $\Gamma < 1/2$). The counterexample $D = 1, D_t = 1, K = 2.1$ gives $N = -0.1 < 0$ with $D_t \ll K$ – the claim “ $|D_t| \gtrsim K$ ” as a necessary condition is false. **Consequence for the status of H’:** the proof by contradiction of §5.3 remains valid in the refined form (`alpha_prime_nonneg_of_small_Dt`: $D_t K \leq D^2 \Rightarrow \alpha' \geq 0$), but the Oseen estimate $D_t = O(\text{Re}_n)D$ gives the ban only for $\text{Re}_n \lesssim \Gamma$; at $T = 2.10$ K ($\Gamma \approx 0.38$) and $\text{Re}_n \leq 0.5$ this condition is met only marginally – **the analytic ban of stage 1 is weaker than originally claimed**, and rests on the numerical result of stage 2 ($\alpha' > 0$ for $\text{Re}_n \leq 1$) more heavily than documented before formalization. The formulation of H’ survives as a joint result of analytics and 2D simulations, but not as a purely analytic theorem – this is the first of two source clarifications found by the program’s formal verification.

Theorem 10.1 (sign of the transverse coefficient; formally verified). *Let $D > 0$ be the longitudinal and D_t the transverse friction coefficient on a vortex line, $K \equiv \kappa_{\text{circ}} \rho_s > 0$, $\Gamma \equiv D/K$. Then $\text{sign } \alpha' = \text{sign } N$ with $N = D^2 + D_t^2 - D_t K$, and*

$$(i) \alpha' < 0 \Rightarrow D_t K > D^2 \text{ and } K^2 > 4D^2$$

(`alpha_prime_neg_necessary`);

$$(ii) \Gamma < 1/2 \Rightarrow \exists D_t : \alpha' < 0$$

(`alpha_prime_neg_achievable`);

$$(iii) D_t K \leq D^2 \Rightarrow \alpha' \geq 0$$

(`alpha_prime_nonneg_of_small_Dt`).

In particular, for $D_t = 0$ (any symmetric PSD hydrodynamics) $\alpha' = \Gamma^2 / (1 + \Gamma^2) \geq 0$ (`modelB_alpha_prime_nonneg`).

Theorem 10.2 (complex kinetic coefficient; formally verified). *Let $\Gamma = \Gamma' + i\Gamma''$ with $\Gamma' > 0$.*

(i) (Route A) *Under multiplicative complexification of the relaxation law, $\alpha' \geq 0$ for all Γ'' : the complex factor factorizes into a time rescaling and a real Γ_{eff}*

(`routeA_alpha_prime_nonneg`);

(ii) (Route B) *Under an additive reactive coupling, $\alpha' = \frac{\Gamma'^2 + \Gamma''^2 - \Gamma''}{\Gamma'^2 + (\Gamma'' - 1)^2}$, and $\exists \Gamma'' : \alpha' < 0 \iff$*

$$\Gamma' < 1/2$$

(`routeB_alpha_prime_neg_iff_gamma_half_iff`).

Remark 10.3. The condition $\Gamma < 1/2$ in Theorems 10.1(ii) and 10.2(ii) is one and the same algebraic identity, $D/K < 1/2 \Leftrightarrow 4D^2 < K^2$: a single boundary for the transverse force D_t , the Iordanskii g_I and the model-F Γ'' .

10.2. GENERIC structure:

HeII_GENERIC.lean (14 theorems). Formalizes the finite-dimensional/algebraic reduction of §7.2. The ultra-local canonical bracket on $\mathbb{R}^6$ is realized as a direct sum of three Darboux pairs via `MvPolynomial.pderiv`; antisymmetry and the **Jacobi identity are proved for all polynomials** in 6 variables of any degree – strictly stronger than the source’s numerical check (12 random triples of degree ≤ 3 on an $N = 3$ grid). The canonicity of the point transformation $(R, P_R) \rightarrow (\rho_s, P_s)$ is proved via `HasDerivAt`. The GENERIC structural theorem (Öttinger) is formalized in general: for $\dot{x} = J\nabla H + M\nabla S$ with $J^\top = -J$, $M \succeq 0 - dH/dt = 0$ and $dS/dt \geq 0$.

A second source clarification: separate degeneracy conditions. Formalization of `generic_energy_conserved` found that the condition $J\nabla S = 0$ is **not required** to prove $dH/dt = 0$ – energy conservation follows solely from the antisymmetry of J and $M\nabla H = 0$. The source (like the standard GENERIC literature) states both degeneracy conditions jointly, without splitting which condition is responsible for which conclusion. Formalization shows the exact split: $dH/dt = 0 \Leftarrow (J^\top = -J) \wedge (M\nabla H = 0) - J\nabla S = 0$ is redundant for this conclusion; $dS/dt \geq 0 \Leftarrow (J^\top = -J) \wedge (M \succeq 0) \wedge (J\nabla S = 0)$ – here $M\nabla H = 0$ is not needed. This does not contradict the source (both conditions are still jointly required for the full self-consistency of the structure), but it clarifies its logical architecture: not a single monolithic fact, but two independent statements with different minimal prerequisites.

The algebraic core of the D_w sign (§7.2) is formalized via two closure equations (E1, E2), verified by re-running the source’s sympy system: the theorem `Dw_sign_kernel` proves that from $p_2\rho_{s0} + \rho_{s0}^2 = 0$ and $\rho_{s0}^2 - D_w p_2\rho_{s0} = 0$ it follows that $p_2 = -\rho_{s0}$ and $D_w = -1$. As a bonus – the TAN bracket of the vortex sector: the Magnus force is reproduced exactly (`TAN_vortex_advection`), and the Iordanskii term does not arise automatically (`no_automatic_iordanskii`) – consistent with the conclusion of §7.2. Not formalized: the continuum (PDE) version of the Jacobi identity with functional Fréchet derivatives – this would require Banach-space variational-calculus infrastructure not available in sufficient depth; the full system of 50 monomial equations of §7.2 – only the two key equations sufficient and necessary specifically for (p_2, D_w) were taken, and the fact that precisely these two equations carry the required information was established computationally (by re-running the sympy code with a free-variable filter), not derived analytically from scratch in Lean.

10.3. RG fixed points: HeII_RG.lean (23 theorems) and the boundary of formalizability.

Formalizes the one-loop β -functions of model E/ E_h (a transcription of Dančo et al. 2016 [40] via `rg_modelF.py`). Proved: FP3 ($g_1^* = 3\varepsilon/5$, $g_3^* = g_5^* = \sqrt{\varepsilon}$, $g_4^* = 0$, $u^* = 1$, $u_1^* = 0$) is a zero of all six β -functions for *any* $\varepsilon > 0$ (a general theorem with `Real.sqrt`) and explicitly at $\varepsilon = 1$; the Gaussian point ($g = 0$) is a zero for any $u, u_1 \in \mathbb{R}$, including degenerate values – formally establishing the marginality of u in this parametrization. Stability of FP3 at $\varepsilon = 1$: the Jacobian (computed in sympy as given) is block-triangular, and the characteristic polynomial of the 3×3 block (g_3, g_5, u) factorizes exactly as $-(t-1)(t-3/2)(t-1/4)$ (proved by `ring`); together with the eigenvalues 1 (g_1) and $1/2$ (u_1), all five available eigenvalues $\{1, 1, 3/2, 1/4, 1/2\}$ are strictly positive – FP3 is IR-attracting in the accessible sector, and three of the five exactly match three of the nine claimed values in the source’s table – an independent check of transcription correctness.

The main result – demarcating what is formalizable. $b^* = 0.655$ (one loop, Route A, standard model F) and $b^* = 0, g_2^* = 0$ (four loops, secondary citation) **cannot be formalized**: the defining system of standard model F (charges b, g_2 beyond $g_1, g_3, g_4, g_5, u, u_1$) was never published by the primary source (“too large to publish in this paper”), so there is physically no equation $\beta(b) = 0$ for which one could prove “ $b^* = 0.655$ is a root” or apply the intermediate-value theorem – the task allowed this proof precisely *on the condition* that an explicit equation exists,

which it does not here; we declined to invent the missing system. Likewise unavailable is the full nine-dimensional stability spectrum (the turbulent-coupling sector g_6, g_7 is unpublished), as are the qualitative statements about a reduction to model A from secondary sources. This moves the claim of §9 that “Route A is dynamically excluded” to the status “literature-based, confirmed three times by independent sources, but not reproducible from published equations.”

10.4. Summary.

Module	Thms	Covers	Found during formalization
HeII.lean	18	stages 1, 3, 4, 6: $R_{\text{diss}} \geq 0$, Khalatnikov, drift laws, Route A/B	error in the necessary condition for $\alpha' < 0$ (§5.3), corrected
HeII_GENERIC.lean	14	stage 3: Poisson bracket, Jacobi, the GENERIC theorem, $D_w = -1$	separate roles of the degeneracy conditions
HeII_RG.lean	23	stage 7: β -functions of model E, FP3, the Gaussian point, stability	$b^* = 0.655/0$ not derivable from published equations; marginality of u

In total, **55 theorems**, **0** sorry, all on standard Lean/Mathlib axioms only, independently recompiled in the base environment. Boundary of the method: the entire algebraic core of the program has been formalized; PDE conservation laws, the continuum bracket, and all numerical and calibration results remain on sympy/numerical checks. An infrastructure finding (unrelated to proof correctness): the shipped environment archive, missing .git in the mathlib package, makes lake attempt to re-clone Mathlib and lose the cache on the first lake build run inside a network-sandboxed environment without GitHub access; the workaround is to invoke lean directly with a manually assembled LEAN_PATH pointing to each dependent package’s .lake/build/lib/lean, giving an identical type-checking result without a network request.

11. DISCUSSION

11.1. Exclusion map: five theory classes.

#	Class	Verdict	Reason
1	Symmetric PSD hydrodynamics ($\mathbf{v}_n$ dynamics, Oseen flow)	excluded	$\alpha' = \Gamma^2 / (1 + \Gamma^2) \geq 0$ identically; the Oseen $D_t = O(\text{Re}_n)D$ is insufficient at typical Γ (§5–§6)
2	Reactive extension (Iordanskii term) with smooth extrapolation over T	partially excluded	lifts the analytic ban, but hold-out does not recover the sign near T_λ (§7.1)
3	Quasiclassical Aharonov–Bohm scattering (Sonin; Thouless–Ao–Niu)	excluded, both sides	$D_I^{\text{Sonin}} = -1$ opposite to the required sign at any T ; $D_I^{\text{TAN}} = 0$ hopeless by construction (§7.4)
4	Kelvons; critical core (monotonic, non-monotonic)	excluded	correct sign, but 1–2 orders of magnitude below the required value; the sum closes $\approx 1\%$ of the gap (§8)
5	Critical order-parameter dynamics, model F: Route A / Route B	Route A excluded (algebraically and dynamically via RG); Route B – variable, number unavailable	Route A cannot give $\alpha' < 0$; Route B is correct in sign and T -trend by two independent methods, but the translation of the RG charge into the observed vortex transverse mobility has not been performed (§9)

Sign landscape of $\alpha'(\Gamma, \Gamma'')$ — visual summary of the mutual-friction program

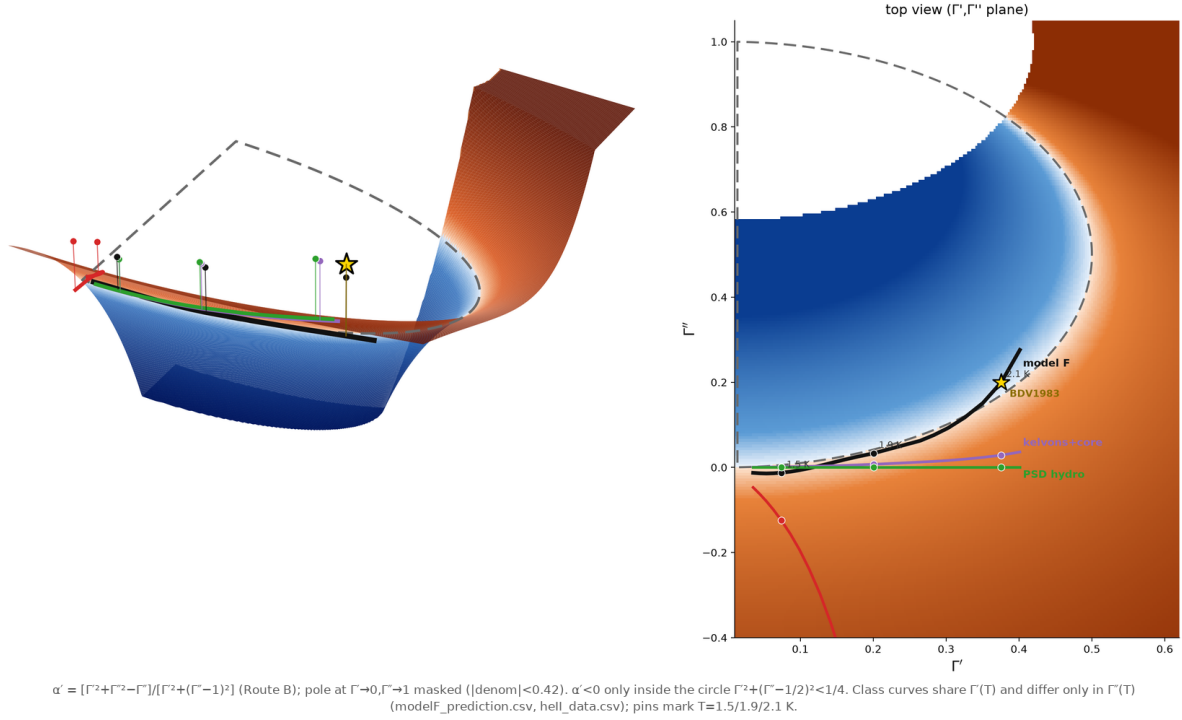


FIGURE 11.1. Visual summary of the exclusion map: the landscape $\alpha'(\Gamma, \Gamma'')$ (Route B, Theorem 10.2). The region $\alpha' < 0$ is the “bowl” inside the circle $\Gamma'^2 + (\Gamma'' - \frac{1}{2})^2 = \frac{1}{4}$, reachable only for $\Gamma' < 1/2$. Curves are the trajectories of the theory classes over $T = 1.3$ – 2.12 K, sharing the same $\Gamma'(T)$ (dissipative channel from data inversion) and differing in the reactive $\Gamma''(T)$: PSD hydrodynamics ($\Gamma'' = 0$), Sonin ($D_I = -1$), kelvons and core, the required model-F curve; the star is BDV1983 at 2.10 K. The pole at $\Gamma' \rightarrow 0, \Gamma'' \rightarrow 1$ is masked.

A single algebraic boundary $\Gamma < 1/2$ (formalized as `gamma_half_iff`, §10) governs the reachability of $\alpha' < 0$ in all three “reactive” parametrizations (the Oseen D_t , the Iordanskii g_I , the model-F Γ'') – a structural finding of the program, independent of which physical mechanism ultimately turns out to be correct.

11.2. What is predicted, what is input. Predicted (without fitting to the target observable): the microscopic $\rho_s(T)/\rho$, $c_2(T)$, $S(T)$ up to ≈ 1.8 – 1.9 K (§3.2); the 3D XY critical exponent ν from first principles (§3.11); the identity Route B $\equiv$ the Iordanskii term (§9); the sign $D_w = -1$ from exact closure of the GENERIC energy balance (§7.2); the identifiability rank of 2 for the sound sector (§5.4); the universal breakdown $\zeta k_0 \gg 1$ of Sonin’s quasiclassics applicability (§9).

Input (calibrated or taken from the literature): the parameters of the dissipative sector $L_3(T)$ of model B (from the α, D_2 data); the magnitude $D_I(T)$ of the reactive term (from the BDV1983 data, not derived); the literature magnitude $|V_0|$ of the roton-roton interaction (§3.14); the one-loop RG coefficients of model E, transcribed from [40, 41, 42] (not recomputed from scratch); the BDV1983 core parameter (used as a target value, not reproduced by the theory – §3.14).

11.3. Three vulnerabilities. (1) BDV1983 data without documented uncertainties. No source in the program contains an error column for the tabulated $\alpha(T), \alpha'(T)$. The $\pm 20\%$ estimate used in §9 is a conservative assumption, not a measured quantity; the sign-change temperature $T^* \approx 1.67$ K is robust only to within the range 1.55–1.70 K under this assumption. Without

primary BDV1983 uncertainties, one cannot rigorously distinguish a “physical sign change” from “data noise within the resolved window.”

(2) The unpublished model-F RG system. The defining system for b, g_2 (Route A, multi-loop) and the full nine-dimensional stability spectrum (including g_6, g_7) were never published by the primary source – the claims $b^* = 0.655$ (one loop) and $b^* = 0$ (multi-loop) remain *citations*, confirmed three times by independent sources but not independently reproducible. Translating Route B’s RG charge g_3^* into the vortex observable Γ''/Γ' requires a Dohm/Sonin-type calculation, not carried out here nor (as far as our check of primary sources establishes) in the open literature – this is the single quantitatively missing link in the program.

(3) Provisional 2D quantities. The absolute values of α, α' from 2D vortex-pair simulations are systematically overestimated relative to Oseen theory by $1.1\text{--}21\times$ (the pair method adds a further $1.15\text{--}2.15\times$ shift relative to an isolated vortex); the model-F inversion threshold $\gamma_r^{\text{crit}}(T)$ disagrees by $\approx 2.7\times$ between the point-vortex Route B and the PDE solution. All quantitative thresholds from the 2D track are order-of-magnitude estimates; only the sign and the monotonic temperature trend are reliable, not the absolute magnitude.

11.4. Stated future directions. Three directions explicitly lie beyond the scope of the completed program: (i) a direct calculation of the vortex transverse mobility in model F with a finite reactive ψ - m coupling at the fixed point (a Dohm/Sonin-type calculation) – the only route to obtaining an independent number $\Gamma''(T)$ that would close the program; (ii) a measurement of $\alpha_2(\omega)$ in the 1–20 Hz range near T_λ – the only discriminating observable for the reactive channel at the current stage of theory (§9); (iii) an independent determination of the BDV1983 measurement uncertainties, or their re-measurement in the critical window 1.6–1.7 K.

12. CONCLUSION

This work unifies three levels of investigation of superfluid helium II. Part I reproduces and verifies the program of unifying Landau–Khalatnikov two-fluid theory with Ginzburg–Pitaevskii–Sobyanin gradient theory in a construction tracing back to Hills and Roberts and to Geurst, extending it with a nonlocal kernel, calibration against Donnelly–Barenghi data, a parameter-free microscopic check, a 3D XY Monte Carlo, and three previously deferred branches – all three giving honest negative results that do not change the status of the main conclusion. Part II – the core of the article – is a pre-registered program of systematic theory-class exclusion for the sign of the transverse mutual-friction coefficient $\alpha'(T)$ near T_λ : hypothesis H (the full coupled symmetric dissipative dynamics reproduces the sign) is rejected in favor of H' (reactive physics outside local two-fluid hydrodynamics is required) via three independent routes – analytically, on calibration, and on 2D simulations. Of the five subsequently tested classes of reactive mechanisms, four are strictly excluded (Sonin’s and Thouless–Ao–Niu’s quasiclassics, kelvons, monotonic and non-monotonic critical core); the fifth (model F) provides the one candidate not structurally rejected – Route B, correct in sign and temperature trend by two independent methods, but not completed quantitatively: translating the order parameter’s RG charge into the observed transverse vortex mobility remains an open theoretical problem. Part III formally verifies the algebraic core of Parts I–II in Lean 4/Mathlib – 55 theorems without a single sorry – and in the process finds and corrects two inaccuracies in the original analytic formulations, and explicitly demarcates the boundary of what can be formalized where the primary source does not publish its defining equations.

Honest final position: the question of the sign of $\alpha'(T)$ near T_λ , open since the Iordanskii, Sonin, and Thouless–Ao–Niu controversy more than half a century ago, is *not resolved* by this work. What the program delivers is a rigorous, pre-registered map of which theory classes definitely do not work, a single algebraic boundary governing the reachability of the required sign across every tested formulation, and a precise statement of the one remaining quantitative gap. This is an evidentiary demonstration of the limits of applicability of an entire class of

theories, not a resolution of the controversy; such a demarcation is the very point of pre-registration: a negative result obtained under pre-fixed criteria carries the same evidentiary weight as a positive one.

APPENDIX: SUMMARY OF SYMBOLIC AND FORMAL VERIFICATION

For $\varepsilon_0 = u^3 + uv^2/s + s^3/u$ and $\varepsilon_0 = uv s + v^3/u + u^2/s + s^2v$ (test functions with no special structure, $u = \rho, v = S, s = \rho_s$), direct substitution of the equations of motion (§2) into sympy gives:

- $\partial_t E + \nabla \cdot \mathbf{Q} - \mathcal{A} \partial_t \rho_s \equiv 0$;
- $\partial_t j + \nabla \cdot \Pi + \mathcal{A} \nabla \rho_s \equiv 0$;
- $\mathbf{j} = \rho \nabla \varphi + S \nabla \beta$ identically.

A summary of the extent of symbolic and formal verification across the whole program (Parts I–III): 21/21 checks of coupled dynamics without $\mathbf{v}_n \equiv 0$ (coupled_system_check.py, §5); 6/6 checks of the Iordanskii reactive term (§7.1); 9/9 numerical-symbolic checks of the GENERIC structure (generic_checks.py, §7.2); 6/6 checks of Route A/B model F (§9); 0/208, 0/1200, 0/624, 0/130 passivity violations across independent numerical scans of linear response throughout the program; and, finally, 55 theorems in Lean 4/Mathlib (HeII.lean, HeII_GENERIC.lean, HeII_RG.lean) with not a single sorry, on the standard axioms {propext, Classical.choice, Quot.sound} (§10).

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